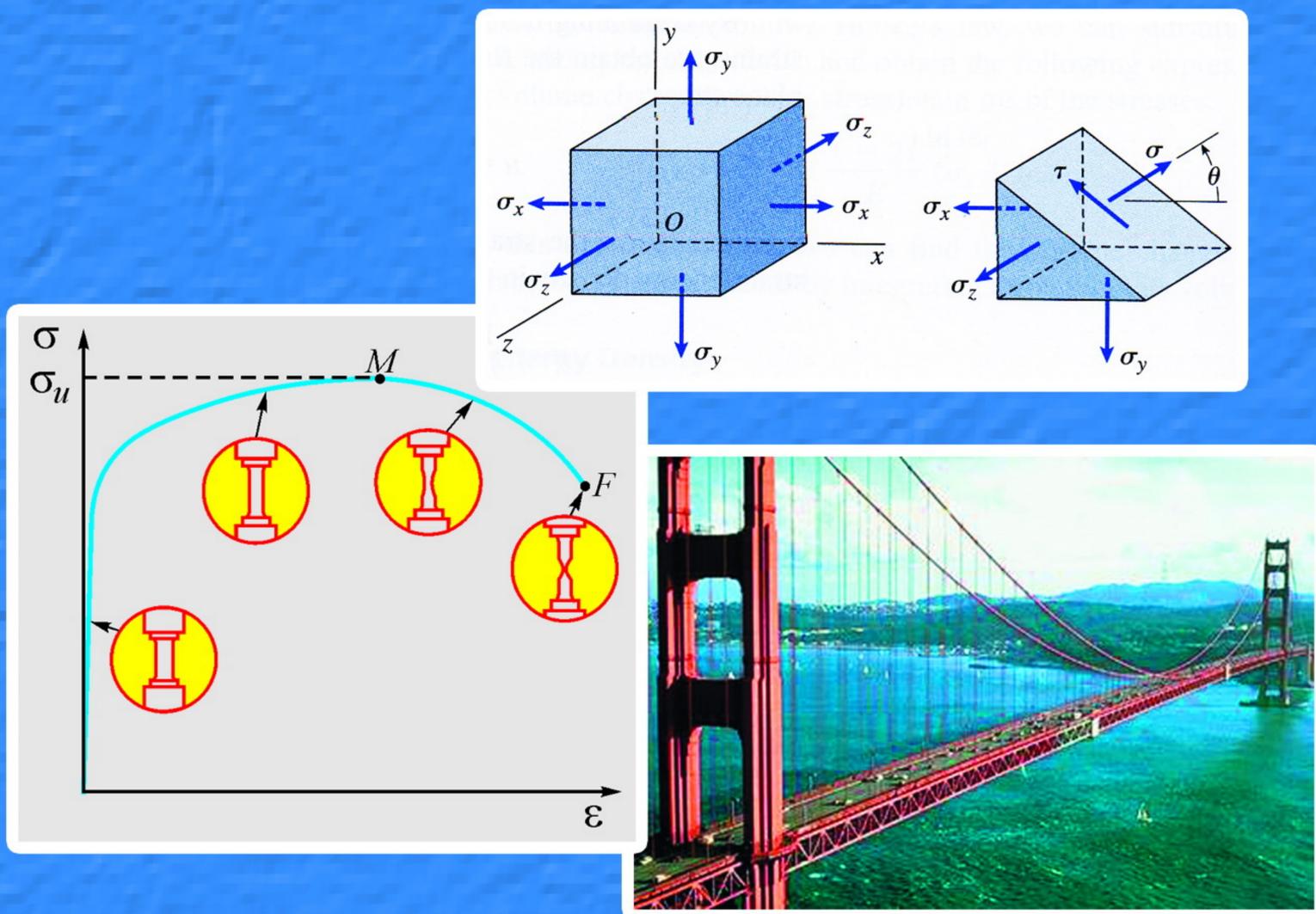


**V.F. Demenko**

**Series: ENGINEERING EDUCATION**

# **EXAMPLES AND PROBLEMS IN MECHANICS OF MATERIALS**



MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE  
INSTITUTE OF INNOVATIONS AND CONTENTS OF EDUCATION

NATIONAL AEROSPACE UNIVERSITY  
"KHARKIV AVIATION INSTITUTE"

SERIES: ENGINEERING EDUCATION

VLADISLAV DEMENKO

# EXAMPLES AND PROBLEMS IN MECHANICS OF MATERIALS

## STRESS-STRAIN STATE AT A POINT OF ELASTIC DEFORMABLE SOLID

EDITOR-IN-CHIEF YAKIV KARPOV

Recommended by the Ministry of Education and Science of Ukraine  
as teaching aid for students of higher technical  
educational institutions

KHARKIV 2010

УДК: 539.319(075.8)  
UDK: 539.319(075.8)

**Деменко В.Ф. Задачі та приклади з механіки матеріалів. Напружено-деформований стан в околі точки пружно-деформованого твердого тіла/ В.Ф. Деменко. – Х.: Нац. аерокосм. ун-т “Харк. авіац. ін-т”, 2009. – 292 с.**

**Examples and Problems in Mechanics of Materials. Stress-Strain State at a Point of Elastic Deformable Solid/ V. Demenko. – Kharkiv: National Aerospace University “Kharkiv Aviation Institute”, 2009. – 292 p.**

ISBN 978-966-662-208-5

Посібник містить важливі розділи дисципліни “Механіка матеріалів” для бакалаврів напрямів “Авіа-та ракетобудування” та “Інженерна механіка”. Починаючи з викладення основних понять цієї фундаментальної загальноінженерної дисципліни, таких, як механічне напруження, деформація, пружне деформування та інші, він охоплює всі основні практичні проблеми теорії напружено-деформованого стану на рівні, достатньому для бакалаврів авіа- і ракетобудування. Вони стосуються в першу чергу визначення діючих напружень, деформацій, а також потенціальної енергії пружної деформації в околі точки пружно-деформованого твердого тіла за умов зовнішнього навантаження різноманітних конструктивних елементів, починаючи від простих деформацій: розтягу - стиску, кручення, плоского гнуття. При розв’язанні прикладів і задач використано Міжнародну систему одиниць (СИ).

Для практичної інженерної підготовки студентів, які навчаються за напрямами “Авіа- і ракетобудування” та “Інженерна механіка”, а також студентів спеціальності “Прикладна лінгвістика” при вивченні англійської мови (технічний переклад). Може бути корисним тим студентам, що готуються до стажування в технічних університетах Європи та США, а також іноземним громадянам, які навчаються в Україні.

Іл. 402. Табл. 27. Бібліогр.: 24 назви

The course-book includes important parts of Mechanics of Materials as an undergraduate level course of this fundamental discipline for Aerospace Engineering and Mechanical Engineering students. It defines and explains the main theoretical concepts (mechanical stress, strain, elastic behavior, etc.) and covers all the major topics of stress-strain state theory. These include the calculation of acting stresses, strains and elastic strain energy at a point of elastic deformable solid under different types of loading on structural members, starting from such simple deformations as tension-compression, torsion, and plane bending. All calculations in the examples and also the data to the problems are done in the International System of Units (SI).

Intended primarily for engineering students, the book may be used by Applied Linguistics majors studying technical translation. It will be helpful for Ukrainian students preparing to continue training at European and U.S. universities, as well as for international students being educated in Ukraine.

Illustrations 402. Tables 27. Bibliographical references: 24 titles

Рецензенти: д-р фіз-мат. наук, проф. П.П. Лепіхін,  
д-р техн. наук, проф. О.Я. Мовшович,  
д-р техн. наук, проф. В.В. Сухов

Reviewed by: Doctor of Physics and Mathematics, Professor P. Lepikhin,  
Doctor of Technical Sciences, Professor O. Movshovich,  
Doctor of Technical Sciences, Professor V. Sukhov

Гриф надано Міністерством освіти і науки України  
(лист № 1.4/18-Г-665 від 07.08.06 р.)  
Sealed by the Ministry of Education and Science of Ukraine  
(letter № 1.4/18-G-665 dated 07.08.06)

ISBN 978-966-662-208-5

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## Introduction to Mechanics of Materials and Theory of Stress-Strain State

*Mechanics of materials* is a branch of applied mechanics that deals with the behavior of deformable solid bodies subjected to various types of loading. Another name for this field of study is *strength of materials*. Rods with axial loads, shafts in torsion, beams in bending, and columns in compression belong to the class of deformable solid bodies.

*In mechanics of materials, the general aim is to calculate the stresses, strains, and displacements* in structures and their components under external loading. If we can find these quantities for all the values of applied loads up to the limiting loads that cause failure, we will have a complete picture of the mechanical behavior of these structures or their components. Understanding the mechanical behavior of all types of structures is essential to design airplanes, buildings, bridges, machines, engines able to withstand an applied loads without failure. *In mechanics of materials we will examine the stresses and strains inside real bodies*, that is, bodies of finite dimensions being deformed under loads. To determine stresses and strains, we use the physical properties of materials as well as numerous theoretical laws and concepts, beginning from the fundamental laws of *theoretical mechanics* whose subject *deals primarily with the forces and motions associated with particles and rigid bodies*.

In mechanics of materials, the most fundamental concepts are *stress* and *strain*. These concepts can be illustrated in their most elementary form by considering a *prismatic bar* subjected to *axial forces*. A prismatic bar is a straight structural member having a constant cross section throughout its length; an axial force is a load directed along the axis of the member, resulting either in its tension or compression. Other examples are the members of a bridge truss, connecting rods in automobile engines, columns in buildings.

Normal and shear stresses in beams, shafts, and rods can be calculated from the basic formulas of mechanics of materials. For instance, the stresses in a beam are given by the *flexure* and *shear formulas* ( $\sigma(z) = M_y z / I_y$  and  $\tau(z) = Q_z S_y^* / (b(z) I_y)$ ), and the stresses in a shaft are given by the *torsion formula* ( $\tau(\rho) = M_x \rho / I_\rho$ ). However, the stresses calculated from these formulas act on *cross sections* of the members, while sometimes larger stresses occur on *inclined sections*. The principal topics of this course-book will deal with the *states of stress* and *strain* at points located *on inclined, or oblique, sections*. The components of stressed and strained states also depend upon the position of the point in a loaded body.

The discussions will be limited mainly to *two-dimensional*, or *plane stress* and *plane strain*. The formulas derived and graphic techniques are helpful in analyzing the *transformation of stress and strain* at a point under various types of loading. The graphical technique will help us to gain a stronger understanding of the stress variation around a point. Also, the transformation laws will be established to obtain an important relationship between  $E$ ,  $G$ , and  $\nu$  for linearly elastic materials.

We will derive expressions for the normal and shear stresses acting on inclined sections in both *uniaxial stress* and *pure shear*. In the case of uniaxial stress, we will show that the maximum shear stresses occur on planes inclined at  $45^\circ$  to the axis, whereas the maximum normal stresses occur on the cross sections. In the case of pure shear, we will find that the maximum tensile and compressive stresses occur on  $45^\circ$  planes. Similarly, the stresses on inclined sections cut through a beam may be larger than the stresses acting on a cross section. To calculate these stresses, we need to determine the stresses acting on inclined planes under a more general stress state known as *plane stress*.

In our discussions of plane stress, we will use infinitesimally small *stress elements* to represent the state of stress at a point in a deformable solid. We will begin our analysis by considering an element whose stresses are known, and then we will derive the *transformation relationships* to calculate the stresses acting on the sides of an element oriented in a different direction.

In stress analysis, we must always keep in mind that *only one intrinsic state of stress exists at a point* in a stressed body, regardless of the orientation of the element being used to portray that state of stress. When we have two elements with different orientations at the same point in the body, the stresses acting on the faces of the two elements are different, but they still represent the same state of stress, namely, the stress at the point under consideration.

The concept of stress is much more complex than vectors are, and in mathematics stresses are called *tensors*. Other tensor quantities in mechanics are *strains* and *moments of inertia*.

When studying stress-strain theory, our efforts will be divided naturally into two parts: first, understanding the logical development of the concepts, and second, applying those concepts to practical situations. The former will be accomplished by studying the derivations and examples that appear in each chapter, and the latter will be accomplished by solving the problems at the ends of the chapters.

In keeping with current engineering practice, this book utilizes only International System of Units (SI).

# Chapter 1 Concepts of Stress and Strain in Deformable Solid

## 1.1 Definition of Stress

A body subjected to *external forces* develops an associated system of *internal forces*. To analyze the strength of any structural element it is necessary to describe the *intensity of those internal forces*, which represents a particularly significant quantity.

Consider one of the isolated segments of a body in equilibrium under the action of a system of forces, as shown in Figs. 1.1 and 1.2. An element of area  $\Delta A$ , positioned on an interior surface passing through a point  $O$ , is acted upon by force  $\Delta F$ . Let the origin of the coordinate axes be located at  $O$ , with  $x$  normal and  $y, z$  tangent to  $\Delta A$ . Generally  $\Delta F$  does not lie along  $x, y$ , or  $z$ . Components of  $\Delta F$  parallel to  $x, y$ , and  $z$  are also indicated in the figure. The *normal stress*  $\sigma$  (sigma) and the *shear, or shearing stress*,  $\tau$  (tau) are then defined as

$$\sigma_{xx} = \sigma_x = \lim_{\Delta A \rightarrow 0} \frac{\Delta F_x}{\Delta A} = \frac{dF_x}{dA}, \tag{1.1}$$

$$\tau_{xy} = \lim_{\Delta A \rightarrow 0} \frac{\Delta F_y}{\Delta A} = \frac{dF_y}{dA}, \quad \tau_{xz} = \lim_{\Delta A \rightarrow 0} \frac{\Delta F_z}{\Delta A} = \frac{dF_z}{dA}.$$

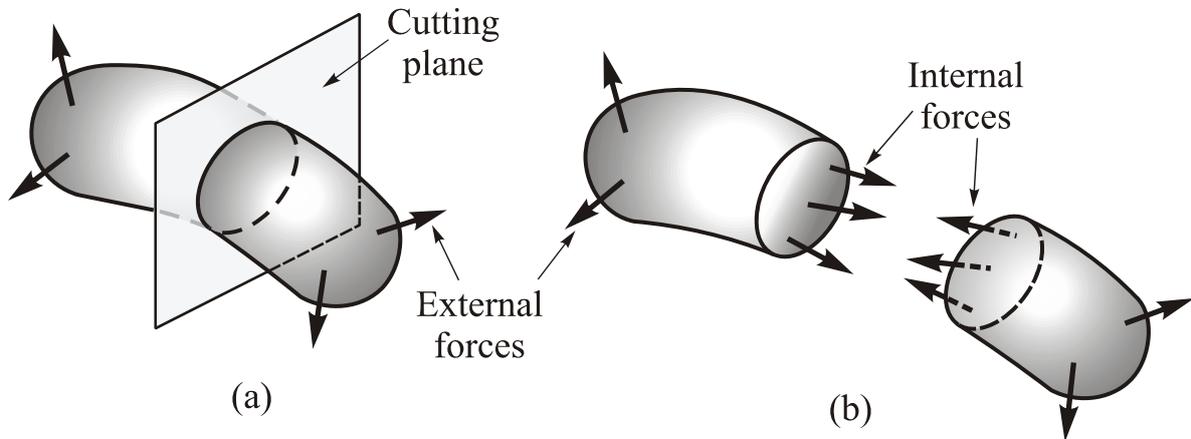


Fig. 1.1 Application of the method of sections to a body under external loading

These relations represent the stress components at the point  $O$  to which area  $\Delta A$  is reduced in the limit.

The primary distinction between normal and shearing stress is one of direction. From the foregoing we observe that two indices are needed to denote the components of stress. For the normal stress component the indices are identical, while for the shear stress component they are mixed. The two indices are given in double subscript notation: the

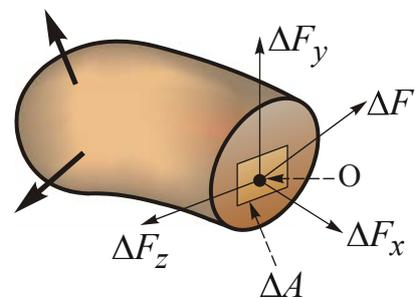


Fig. 1.2 Components of an internal force  $\Delta F$  acting on a small area centered at point  $O$

first subscript indicates the direction of a normal to the plane, or face, on which the stress component acts; the second subscript relates to the direction of the stress itself. Repetitive subscripts will be avoided, so that the normal stress will be designated  $\sigma_x$ , as seen in Eqs. (1.1). **Note that a plane is defined by the axis normal to it; for example, the  $x$  face is perpendicular to the  $x$  axis.**

The limit  $\Delta A \rightarrow 0$  in Eqs. (1.1) is, of course, an idealization, since the area itself is not continuous on an atomic scale. Our consideration is with the *average stress* on areas *where size*, while small as compared with the size of the body, *are large* as compared with the distance between atoms in the solid body. Therefore stress is an adequate definition for engineering purposes. **Note that the values obtained from Eqs. (1.1) differ from point to point on the surface as  $\Delta F$  varies.** The components of stress depend not only upon  $\Delta F$ , however, but also upon the orientation of the plane on which it acts at point  $O$ . Thus, even at a specified point, the stresses will differ as different planes are considered. The complete description of stress at a point therefore requires the specification of stress on all planes passing through the point.

The units of stress ( $\sigma$  or  $\tau$ ) consist of units of force divided by units of area. In SI units, stress is measured in *newtons per square meter* ( $N/m^2$ ) or in *pascals* (Pa). Since the pascal is a very small quantity, the megapascal (MPa) and gigapascal (GPa) are commonly used.

## 1.2 Components of Stress

In order to be able to determine stresses on an infinite number of planes passing through a point  $O$  (Fig. 1.1), thus defining the *state of stress* at that point, we need only specify the stress components on three mutually perpendicular planes passing through the point. These planes, perpendicular to the coordinate axes, contain three sides of an *infinitesimal cubic element (stress element)*.

This *three-dimensional state of stress* acting on an isolated element within a body is shown in Fig. 1.3. Stresses are considered to be identical at points  $O$  and  $O'$  and are uniformly distributed on each face. They are indicated by a single vector acting at the center of each face. We observe a total of nine components of stress that compose three groups of stresses acting on the mutually perpendicular planes passing through  $O$ . This representation of state of stress is called a *stress tensor*. It is a *tensor of second rank*, requiring two indices to identify its elements or components. **Note: a vector is a tensor of first rank; a scalar is a tensor of zero rank.**

Let us consider the property of shear stress from an examination of the equilibrium of forces acting on the cubic element shown in Fig. 1.3. It is clear that the first three of equations of equilibrium of a body in space (the sum of all forces acting upon a body in any direction must be 0) are satisfied. Taking next three equations of equilibrium as moments of the  $x, y$ - and  $z$ -directed forces about point  $O$ , we find that  $\sum M_z = 0$  results in

$$(-\tau_{xy} dy dz) dx + (\tau_{yx} dx dz) dy = 0,$$

from which

$$\tau_{xy} = \tau_{yx}. \quad (1.2)$$

Similarly, from  $\sum M_x = 0$  and  $\sum M_y = 0$ , we obtain  $\tau_{xz} = \tau_{zx}$  and  $\tau_{yz} = \tau_{zy}$ .

The subscripts defining the shear stresses are commutative, and the *stress tensor is symmetric*. This means that *each pair of equal shear stresses acts on mutually perpendicular planes*. Because of this, no distinction will hereafter be made between the stress components  $\tau_{xy}$  and  $\tau_{yx}$ ,  $\tau_{xz}$  and  $\tau_{zx}$ , or  $\tau_{yz}$  and  $\tau_{zy}$ . It is verified that the foregoing is valid even when stress components vary from one point to another.

We shall employ here a *sign convention* that applies to both normal and shear stresses and that is based upon the relationship between the direction of an *outward normal* drawn to surface and the direction of the stress components on the same surface. When *both* the outer normal and the stress component point in a positive (or negative) direction relative to the coordinate axes, the *stress is positive*. When the normal points in a positive direction while the stress points in a negative direction (or vice versa), the *stress is negative*.

Accordingly, tensile stresses are always positive and compressive stresses always negative.

It is clear that the same sign and the same notation apply no matter which face of a stress element we choose to work with. Figure 1.3 depicts positive normal and shear stresses. This sign convention for stress will be used throughout the text.

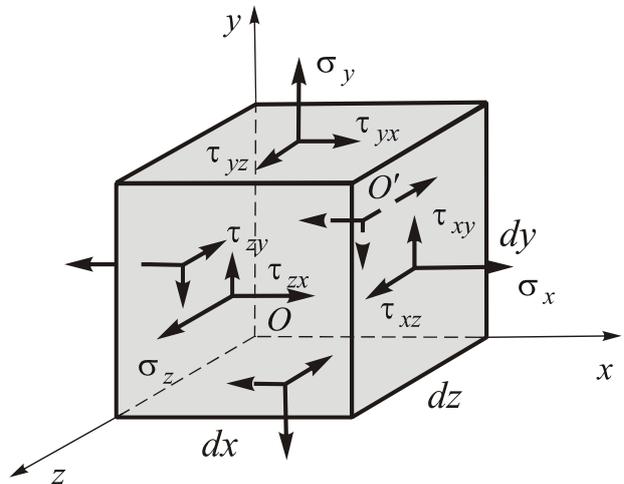


Fig. 1.3 Three-dimensional state of stress

Consider the projection on the  $xy$  plane of a thin element and assume that  $\sigma_x, \sigma_y, \tau_{xy}$  do not vary throughout the thickness and that other stress components are zero. When only one normal stress exists, the stress is referred to as a *uniaxial*, or *one-dimensional*, stress (Fig. 1.4a); when only two normal stresses occur, the state of stress is called *biaxial* (Fig. 1.4b). An element subjected to shearing stresses alone (Fig. 1.4c) is said to be in *pure shear*. The combinations of these stress situations, *two-dimensional* stress (Fig. 1.4d), will be analyzed below.

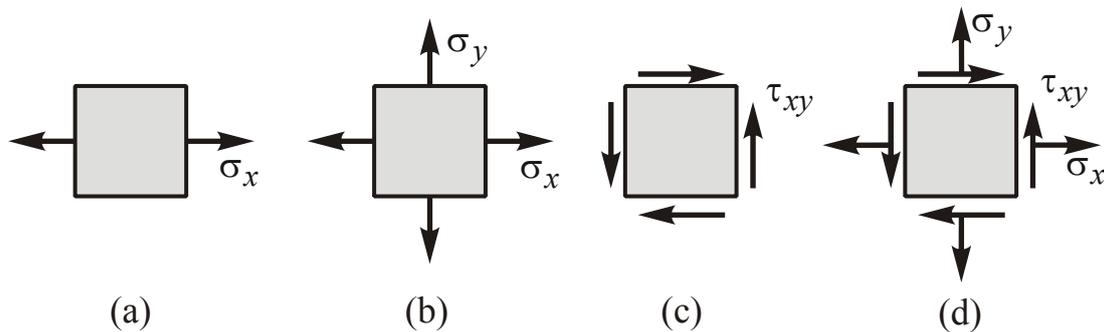


Fig. 1.4 Special cases of state of stress: (a) uniaxial; (b) biaxial; (c) pure shear; and (d) two-dimensional.

### 1.3 Normal Stress

The condition under which the *stress* is *constant* or *uniform at a section* within a body is known as *simple stress*. In many *load-carrying members*, the internal actions on an imaginary cutting plane consist of either only the *axial force* or only the *shear force*. Examples of such elements include *cables*, simple *truss members*, centrally loaded *brace rods* and *bars*, and *bolts*, *pins*, and *rivets* connecting two members. In these bodies, the values of the simple normal stress and shearing stress associated with each action can be approximated directly from the definition of stress and the conditions of equilibrium. However, to learn the "exact" stress distribution, it is necessary to consider the deformations resulting from the particular mode of application of the loads.

Consider, for example, the extension of a *prismatic bar* subject to an axial force  $P$ . A prismatic bar or rod is a straight member having constant cross section throughout its length. The front and top views of such a rod are shown in Fig. 1.5a. To obtain the algebraic expression for the normal stress, we make an imaginary cut (section *a-a*) through the member at right angles to its axis. The *free-body diagram* of the isolated segment of the bar is shown in Fig. 1.5b. The stress is substituted on the cut section as a replacement for the effect of the removed portion in accordance with *method of sections*. The equilibrium of axial

forces requires that  $P = A\sigma$ , where  $A = bh$  is the *cross-sectional area* of the rod. That is, the system of stress distribution in the rod is statically equivalent to the force  $P$ . The *normal stress* is thus

$$\sigma = \frac{P}{A}. \quad (1.3)$$

When the rod is being stretched as shown in the figure, the resulting stress is a *uniaxial tensile stress*; if the direction of the forces is reversed, the rod is in compression and *uniaxial compressive stress* occurs. In the latter case, Eq. (1.3) is applicable only to short members, which are stable to *buckling*.

Equation (1.3) represents the value of the *uniform stress* over the cross section rather than the stress at a specified point of the cross section. When a *nonuniform stress distribution* occurs, then we must deal instead with the *average stress*. A uniform distribution of stress is possible only if three conditions coexist:

1. The axial force  $P$  acts through the *centroid* of the cross section.
2. The rod is straight and made of a *homogeneous material*.
3. The cross section is remote from the ends of the rod, excluding so called *Saint-Venant's zones*.

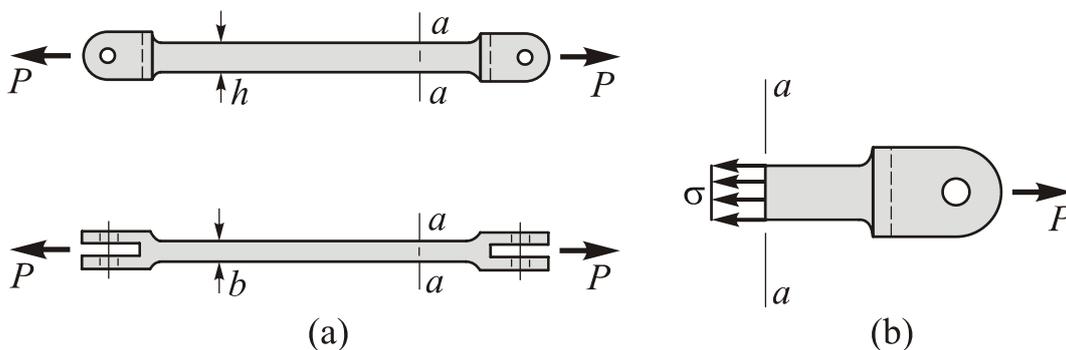


Fig. 1.5 (a) Prismatic bar with clevised ends in tension and (b) free-body diagram of the bar segment

In practice, the force  $P$  is applied to *clevised-forked-ends* of the rod through a connection such as shown in Fig. 1.6a. This joint consists of a *clevis*  $A$ , a *bracket*  $B$ , and a *pin*  $C$ . As the force  $P$  is applied, the bracket and the clevis press against the rivet in *bearing*, and a nonuniform pressure develops against the pin (Fig. 1.6b). The average value of this pressure is determined by dividing the force transmitted by the projected area of the pin into the bracket (or clevis). This is called the *bearing stress*. The bearing stress in the bracket then equals  $\sigma_b = P/(t_1d)$ . Here  $t_1$  is the thickness of the bracket and  $d$  is the diameter of the pin. Similarly, the bearing stress in the clevis is given by  $\sigma_b = P/(2td)$ .

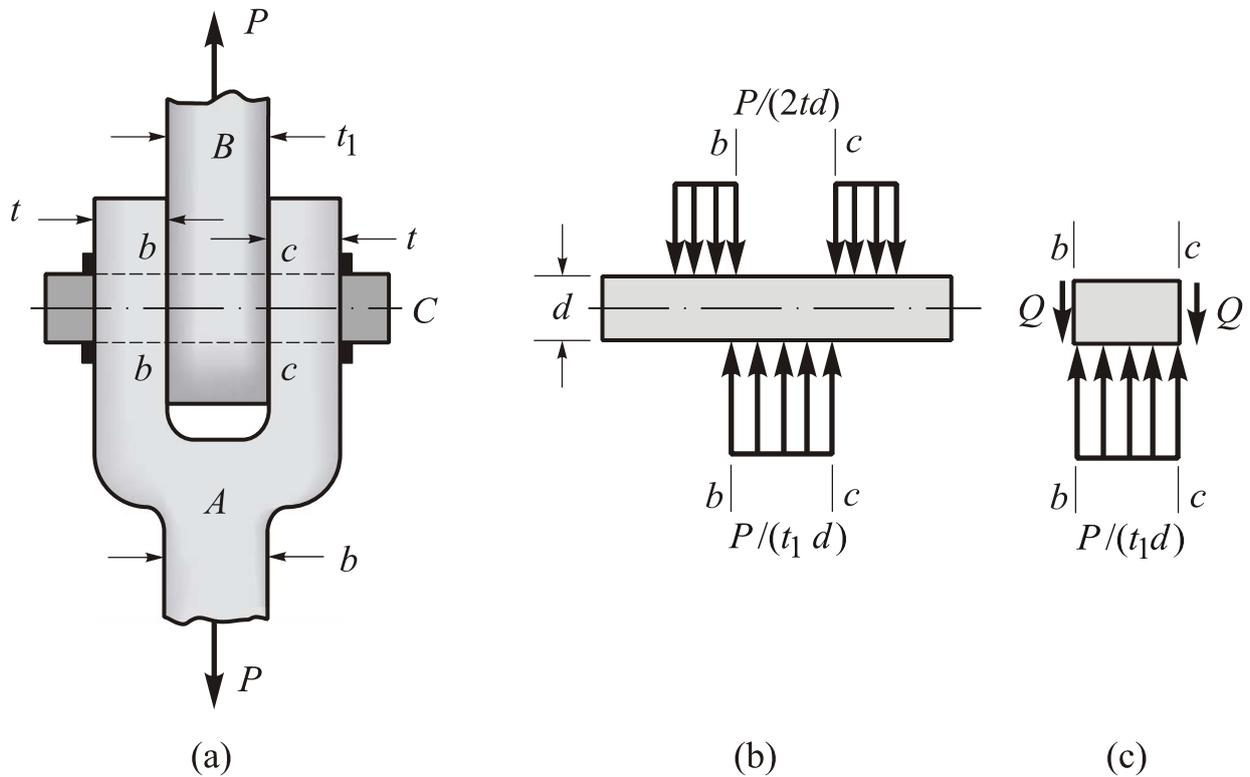


Fig. 1.6 (a) A clevis-pin connection; (b) pin in bearing; and (c) pin in double shear

### 1.4 Average Shear Stress

A *shear(ing) stress* is produced whenever the applied forces cause one section of a body to tend to slide past its adjacent section. An example is shown in Fig. 1.6, where the pin resists the shear across the two cross-sectional areas at  $b-b$  and  $c-c$ . This rivet is said to be in *double shear*. Since the pin as a whole is in equilibrium, any part of it is also in equilibrium. At each cut section, a *shear force*  $Q$  equivalent to  $P/2$ , as shown in Fig. 1.6c, must be developed. Thus the shear occurs over an area parallel to the applied load. This condition is termed *direct shear*.

Unlike normal stress, the distribution of shearing stresses  $\tau$  across a section cannot be taken as uniform. Dividing the total shear force  $Q$  by the cross-sectional area  $A$  over which it acts, we can determine the *average shear stress* in the section:

$$\tau_{av} = \frac{Q}{A}. \quad (1.4)$$

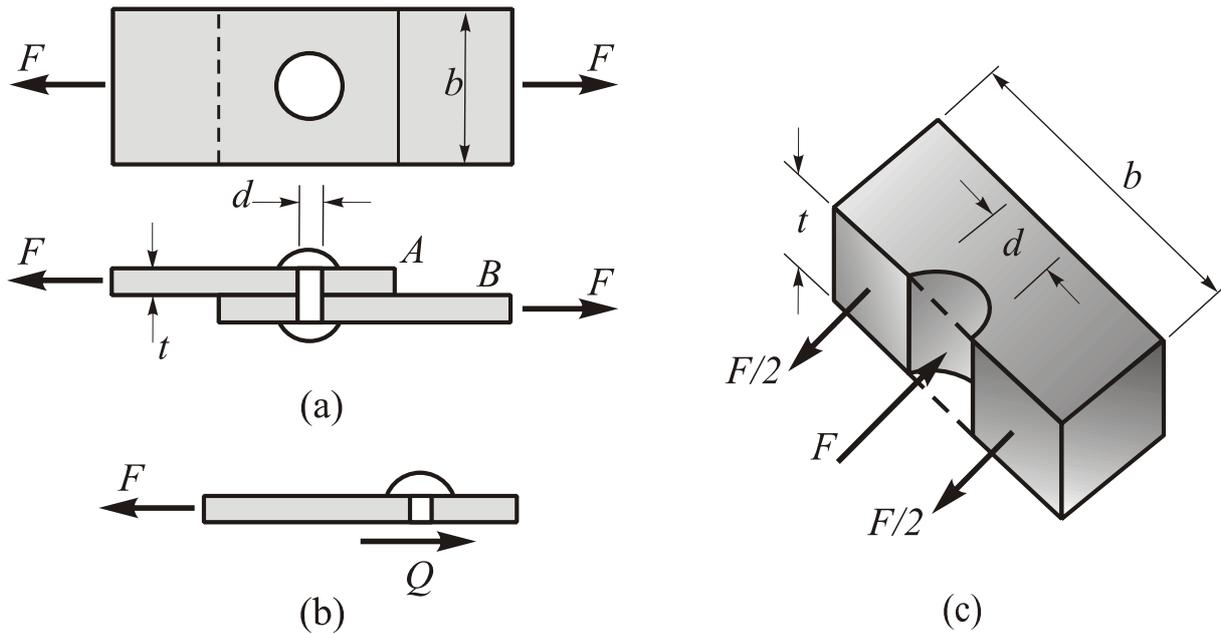


Fig. 1.7 Single shear of a rivet

Two other examples of direct shear are depicted in Figs. 1.7 and 1.8. Figure 1.7a illustrates a connection where plates  $A$  and  $B$  are joined by a rivet. The rivet resists the shear across its cross-sectional area, a case of *single shear*. The shear force  $Q$  in the section of the rivet is equal to  $F$  (Fig. 1.7b). The average shearing stress is therefore  $\tau_{av} = F/(\pi d^2/4)$ .

The rivet exerts a force  $F$  on plate  $A$  equal and opposite to the total force exerted by the plate on the rivet (Fig. 1.7c). The *bearing stress* in the plate is obtained by dividing the force  $F$  by the area of the rectangle representing the projection of the rivet on the plate section. As this area is equal to  $td$ , we have  $\sigma_b = F/(td)$ . On the other hand, the average normal stress in the plate on the section through the hole is  $\sigma = F/[t(b-d)]$ , while on any other section  $\sigma = F/(bt)$ .

Direct shear stresses evidently applied to a plate specimen as shown in Fig. 1.8a. A hole is to be punched in the plate. The force applied to the punch is designated  $P$ . Equilibrium of vertically directed forces requires that  $Q = P$  (Fig. 1.8b). The area resisting the shear force  $Q$  is analogous to the edge of a *coin* and equals  $\pi td$ . Equation (1.4) then yields  $\tau_{av} = P/(\pi td)$ .

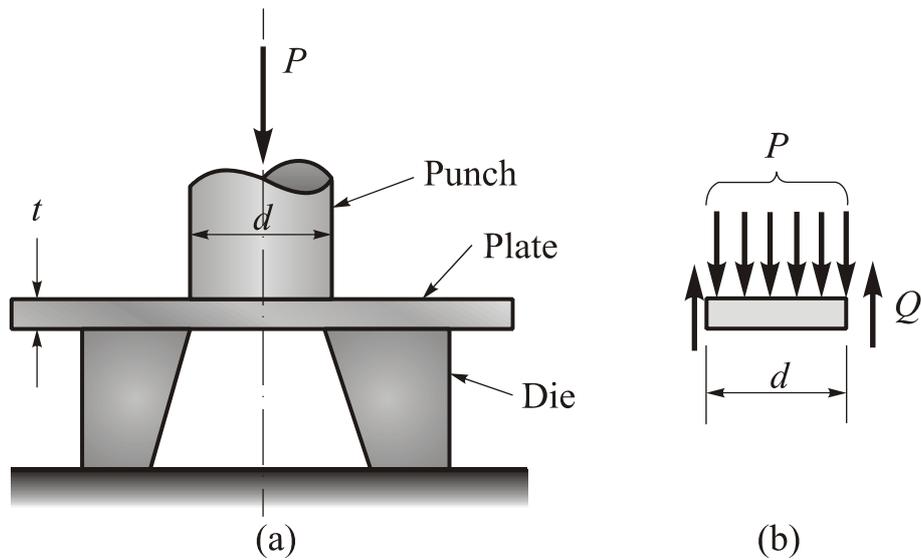


Fig. 1.8 Direct-shear testing in a cutting fixture

## 1.5 Deformations

Consider a body subjected to external forces, as shown in Fig. 1.1a. Owing to the loading, all *points* in the body *are displaced* to new positions. The *displacement* of any point may be a consequence of *deformation*, *rigid-body motion* (translation and rotation), or some *combination of the two*. If the relative positions of points in the body are altered, the body has experienced deformation. If the distance between any two points in the body remains fixed, yet displacement is evident, the displacement is attributable to rigid-body motion. We shall not treat rigid-body displacements because this problem is one of the most important in *theoretical mechanics*. Only *small displacements by deformation*, commonly found in engineering structures, will be considered.

*Extension, contraction, or change of shape* of a body may occur as a result of deformation. In order to determine the actual stress distribution within a member, it is necessary to understand the type of the deformation taking place in that member. Examination of the deformations caused by loading, or by a change in temperature in various members within a structure, makes it possible to compute statically indeterminate forces, i.e. open the statical indeterminacy of structural members.

We shall designate the total *axial deformations* by  $\delta$  (delta). The *components of displacement* at a point within a body in the  $x$ ,  $y$ ,  $z$  directions are denoted by  $u$ ,  $v$ , and  $w$ , respectively. The *strains* resulting from small deformations *are small compared with unity*, and their products (higher-order terms) are omitted. This assumption leads to one of the fundamentals of solid

mechanics, the *principle of superposition*. It is valid whenever the quantity (deformation or stress) to be determined is directly proportional to the applied loads. In such cases, the *total quantity owing to all the loads acting simultaneously on a member may be found by determining separately the quantity due to each load and then combining the results obtained*. The superposition principle permits a complex loading to be replaced by two or more simpler loads.

## 1.6 Definition of Strain

The concept of *normal strain* is illustrated by considering the deformation of a prismatic bar (Fig. 1.9a). The initial length of the member is  $L$ . After application of a load  $F$ , the length increases an amount  $\delta$  (Fig. 1.9b). Defining the *normal strain*  $\varepsilon$  (epsilon) as the *unit change in length*, we obtain

$$\varepsilon = \frac{\delta}{L}. \quad (1.5)$$

A positive sign applies to *elongation*, a negative sign to *contraction*.

The *shearing strain* is the tangent of the total change in angle occurring between two perpendicular lines in a body during deformation. To illustrate, consider the deformation involving a change in shape (*distortion*) of a rectangular plate (Fig. 1.10). Note that the deformed state is shown by the dashed lines in the figure, where  $\theta'$  represents the angle between the two rotated edges. Since the displacements considered are small, we can set the tangent of the angle of distortion equal to the angle. Thus the *shearing strain*  $\gamma$  (gamma) measured in radians, is defined as

$$\gamma = \frac{\pi}{2} - \theta'. \quad (1.6)$$

The shearing strain is *positive* if the right angle between the reference lines *decreases*, as shown in the figure; otherwise, the shearing strain is negative.

When uniform changes in angle and length occur, Eqs. (1.5) and (1.6) yield results of acceptable accuracy. In cases of nonuniform deformation, the strains are defined at a point. This *state of strain* at a point will be discussed below.

Both *normal and shear strains are indicated as dimensionless quantities*. In practice, the normal strains are also frequently expressed in terms of *meter* (or *micrometer*) per meter, while shear strains are expressed in *radians* (or *microradians*). For most engineering materials, strains seldom exceed values of 0.002 or  $2000 \mu$  ( $2000 \times 10^{-6}$ ) in the elastic range.

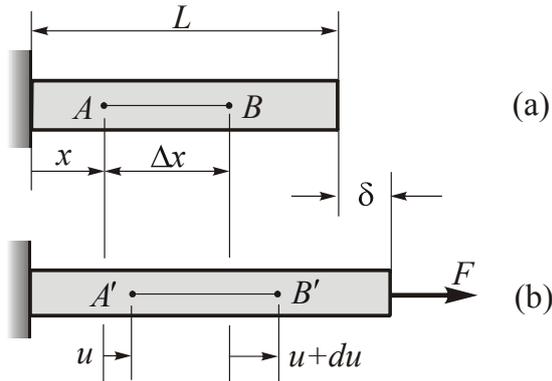


Fig. 1.9 Deformation of a prismatic bar

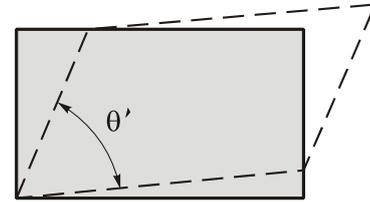


Fig. 1.10 Distortion of a rectangular plate

### 1.7 Components of Strain

When uniform deformation does not occur, the strains vary from point to point in a body. Then the expressions for uniform strain must relate to a line  $AB$  of length  $\Delta x$  (Fig. 1.9a). Under the axial load, the end point of the line experiences displacements  $u$  and  $u + \Delta u$  to become  $A'$  and  $B'$ , respectively (Fig. 1.9b). That is, an elongation  $\Delta u$  takes place. The definition of normal strain is thus

$$\epsilon_x = \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} = \frac{du}{dx} \tag{1.7}$$

In view of the limit, the foregoing represents the strain at a point to which  $\Delta x$  shrinks.

In the case of two-dimensional, or plane, strain, all points in the body, before and after application of load, remain in the same plane. Thus the deformation of an element of dimensions  $dx$ ,  $dy$  and of unit thickness can contain linear strains (Fig. 1.11a) and a shear strain (Fig. 1.11b). For instance, the rate of change of  $u$  in the  $y$  direction is  $\partial u / \partial y$ , and the increment of  $u$  becomes  $(\partial u / \partial y) dy$ . Here  $\partial u / \partial y$  represents the slope of the initially vertical side of the infinitesimal element. Similarly, the horizontal side tilts through an angle  $\partial v / \partial x$ . The partial derivative notation must be used since  $u$  or  $v$  is a function of  $x$  and  $y$ . Recalling the basis of Eqs. (1.6) and (1.7), we can use Fig. 1.11 to come to

$$\epsilon_x = \frac{\partial u}{\partial x}; \quad \epsilon_y = \frac{\partial v}{\partial y}; \quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{1.8}$$

Clearly,  $\gamma_{xy}$  represents the shearing strain between the  $x$  and  $y$  (or  $y$  and  $x$ ) axes. Hence we have  $\gamma_{xy} = \gamma_{yx}$ .

Strains at a point in a rectangular prismatic element of sides  $dx$ ,  $dy$ , and  $dz$  are obtained in a like manner. The *three-dimensional strain* components are  $\varepsilon_x$ ,  $\varepsilon_y$ ,  $\gamma_{xy}$  and

$$\varepsilon_z = \frac{\partial w}{\partial z}; \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}; \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \quad (1.9)$$

where  $\gamma_{xz} = \gamma_{zx}$  and  $\gamma_{yz} = \gamma_{zy}$ . Equations (1.8, 1.9) express the *strain tensor* in a manner like that of the *stress tensor*. If the values of the above strains are known at a point, the increase in size and the change of shape of an element at that point are completely determined.

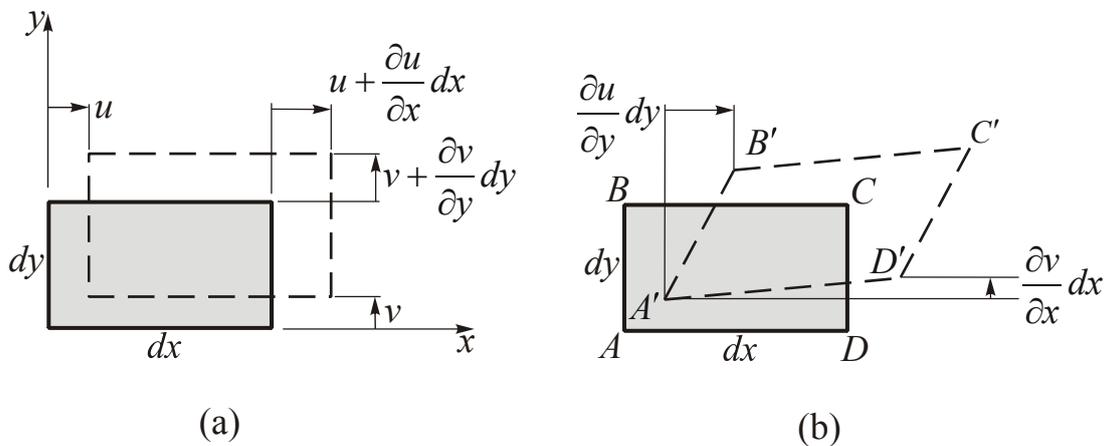


Fig. 1.11 Deformations of an element: (a) linear strain and (b) shear strain

Six strain components depend linearly on the derivatives of the three displacement components. Therefore the strains cannot be independent of one another. Six expressions, known as the *equations of compatibility*, can be derived to show the interrelationships among  $\varepsilon_x$ ,  $\varepsilon_y$ ,  $\varepsilon_z$ ,  $\gamma_{xy}$ ,  $\gamma_{yz}$ , and  $\gamma_{xz}$ . The number of such equations becomes *one* for a two-dimensional problem. *The expressions of compatibility state that the deformation of a body is continuous.* Physically, this means that *no voids are created in the body.* **The approach of the theory of elasticity is based upon the requirement of strain compatibility as well as on stress equilibrium and on the general relationships between stresses and strains.**

In the method of mechanics of materials, basic assumptions are made concerning the distribution of strains in the body as a whole so that the difficult task of solving Eqs. (1.8, 1.9) and of satisfying the equations of compatibility is simplified. The assumptions regarding the strains are based upon the measured strains.

## 1.8 Measurement of Strains (Beginning)

Various mechanical, electrical, and optical systems have been developed for measuring the normal strain on the *free surface* of a member where a *state of plane stress* exists. An extensively used and the most accurate method employs *electrical strain gages*.

Strain gage consists of a grid of fine wire or foil filament cemented between two sheets of treated paper foil or plastic backing (Fig. 1.12a). The backing serves to insulate the grid from the metal surface on which it is to be bonded. Generally, 0.03-mm-diameter wire or 0.003-mm foil filament is used. As the surface is strained, the grid is lengthened or shortened, which changes the electrical resistance of the gage. A bridge circuit, connected to the gage by means of wires, is then used to translate variations in electrical resistance into strains. An instrument used for this purpose is the *Wheatstone bridge*.

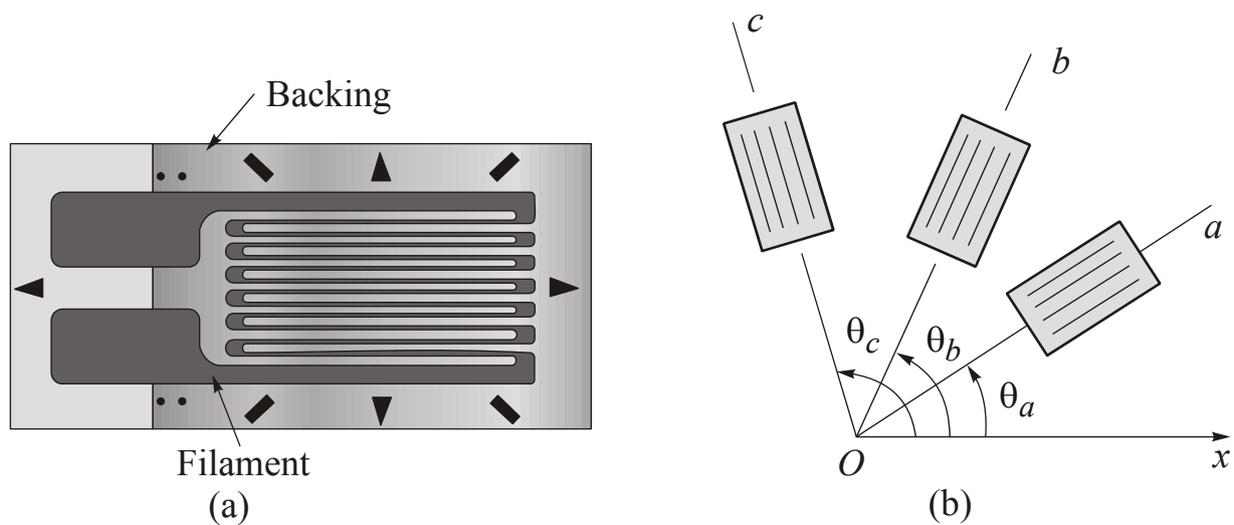


Fig. 1.12 Strain gage

## 1.9 Engineering Materials

In the case of the one-dimensional problem of an axially loaded member, stress-load and strain-displacement relations represent two equations involving three unknown values—stress  $\sigma_x$ , strain  $\epsilon_x$ , and displacement  $u$ . *The insufficient number of available expressions is compensated for by a material-dependent relationship connecting stress and strain*. Hence the loads acting on a member, the resulting displacements, and the mechanical properties of the materials can be associated.

It is necessary to define some important characteristics of commonly used engineering materials (for example, various *metals*, *plastics*, *wood*, *ceramics*, *glass*, and *concrete*). The *tensile test* provides information on material behavior.

An *elastic material* is one which returns to its original (unloaded) size and shape after the removal of applied forces. The elastic property, or so-called *elasticity*, thus excludes *permanent deformation*. Usually the elastic range includes a region throughout which stress and strain are related linearly. This portion of the stress-strain variation ends at a point called the *proportional limit*. Materials having this elastic range are said to be *linearly elastic*.

In the case of *plasticity*, total recovery of the size and shape of a material does not occur. Our consideration in this text will be limited to elastic materials.

A material is said to behave in a *ductile* manner if it can undergo large strains prior to fracture. Ductile materials, which include *structural steel* (a *low-carbon steel* or *mild steel*), many *alloys* of other metals, and *nylon*, are characterized by their ability to yield at normal temperatures. The converse applies to *brittle* materials. That is, a brittle material (for example, *cast iron* and *concrete*) exhibits little deformation before *rupture* and, as a result, fails suddenly without visible deformation. A member that ruptures *is said to fracture*. It should be noted that the distinction between ductile and brittle materials is not so simple. The nature of stress, the temperature, and the rate of loading all play a role in defining the boundary between *ductility* and *brittleness* (so called ductile-to-brittle transition).

Under certain circumstances, the deformation of a material may continue with time while the *load remains constant*. This deformation, beyond that experienced when the material is initially loaded, is called *creep*. On the other hand, a loss of stress is also observed with time even though the strain level remains constant in a member. Such loss is called *relaxation*, it is basically a relief of stress through the mechanism of internal creep. In materials such as *lead*, *rubber*, and certain plastics, creep may occur at ordinary temperatures. Most metals, on the other hand, manifest appreciable creep only when the absolute temperature is roughly 35 to 50 percent of the melting temperature. The rate at which creep proceeds in a given material is dependent not only on temperature but on stress and history of loading as well. In any event, stresses must be kept low in order to prevent intolerable deformations caused by creep.

A *composite* material is made up of two or more distinct constituents. Composites usually consist of a high-strength material (for example, *fibers* made of steel, glass, *graphite*, or *polymers*) embedded in a surrounding material (for example, *resin*, concrete, or nylon), which is termed a *matrix*. Thus a composite material exhibits a relatively large *strength-to-weight ratio* compared with a homogeneous material; composite materials generally have other desirable characteristics and are widely used in various structures, pressure vessels, and machine components.

We assume that materials are *homogeneous* and *isotropic*. A homogeneous solid displays identical properties throughout. If the properties are identical *in all directions* at a point, the material is *isotropic*. A nonisotropic, or *anisotropic*, material displays direction-dependent properties. Simplest among these are those in which the material properties differ in *two* mutually perpendicular directions. A material so described (for example, wood) is *orthotropic*. Mechanical processing operations such as *cold-rolling* may contribute to minor anisotropy, which in practice is often ignored. Mechanical processes and/or *heat treatment* may also cause high (as large as 70 or 105 MPa) internal stress within the material. This is termed *residual stress*. In the cases treated below, *materials are assumed to be entirely free of such stress*.

### 1.10 Allowable Stress and Factor of Safety

To account for *uncertainties* in various aspects of analysis and design of structures—including those related to service loads, material properties, maintenance, and environmental factors — it is of practical importance to select an adequate *factor of safety*. A significant area of uncertainty is connected with the assumptions made in the analysis of stress and deformation. In addition, one is not likely to have sure knowledge of the stresses that may be introduced during the manufacturing and shipment of a part.

The factor of safety is used to provide assurance that the load applied to a member does not exceed the largest load it can carry. *This factor is the ratio of the maximum load the member can sustain under testing without failure to the load allowed under service conditions*. When a linear relationship exists between the load and the stress caused by the load, the factor of safety  $f_s$  may be expressed as

$$\text{Factor of safety} = \frac{\text{maximum usable stress}}{\text{allowable stress}}$$

or

$$f_s = \frac{\sigma_{\max}}{\sigma_{all}}$$

(1.10)

The *maximum usable stress*  $\sigma_{\max}$  represents either the *yield stress* or the *ultimate stress*. The *allowable stress*  $\sigma_{all}$  is the working stress. If the factor of safety used is too low and the allowable stress is too high, the structure may prove weak in service. On the other hand, when the working stress is relatively low and the factor of safety relatively high, the structure becomes unnecessarily heavy and uneconomical.

Values of the factor of safety are usually 1.5 or greater. The value is selected by the designer on the basis of experience and judgment. For the majority of applications, pertinent factors of safety are found in various construction and manufacturing codes.

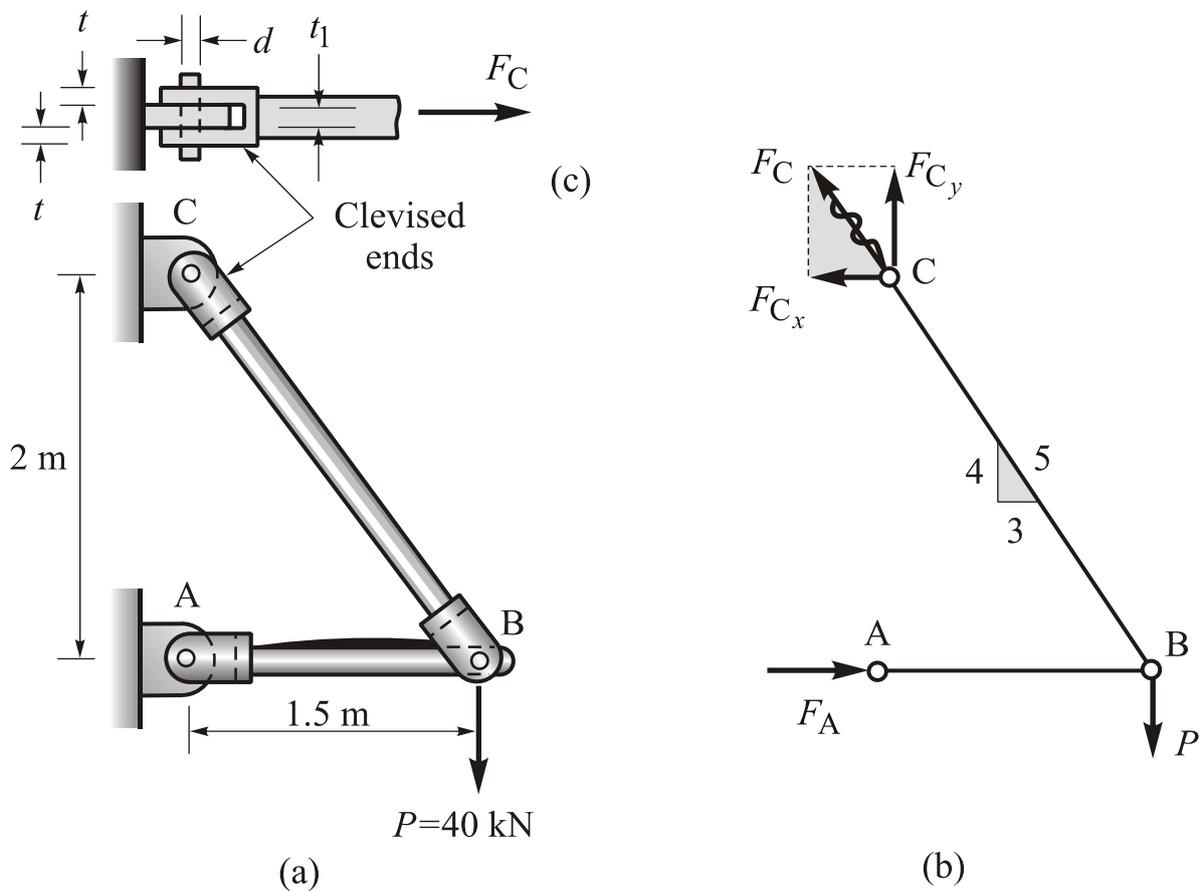
In the field of aeronautical engineering, the *margin of safety* is used instead of the factor of safety. The margin of safety is defined as the factor of safety minus 1, or  $f_s - 1$ .



### EXAMPLES

#### Example 1.1

A pin-connected truss composed of members  $AB$  and  $BC$  is subjected to a vertical force  $P = 40 \text{ kN}$  at joint  $B$  (see figure). Each member is of constant cross-sectional area:  $A_{AB} = 0.004 \text{ m}^2$  and  $A_{BC} = 0.002 \text{ m}^2$ . The diameter  $d$  of all pins is 20 mm, clevis thickness  $t$  is 10 mm, and the thickness  $t_1$  of the bracket is 15 mm. Determine the normal stress acting in each member and the shearing and bearing stresses at joint  $C$ .



**Solution** A free-body diagram of the truss is shown in Fig. b. The magnitudes of the axially directed end forces of members  $AB$  and  $BC$ , which are equal to the support reactions at  $A$  and  $C$ , are labeled  $F_A$  and  $F_C$ , respectively. For computational convenience the  $x$  and  $y$  components of the inclined forces are used rather than the forces themselves. Hence force  $F_C$  is resolved into  $F_{C_x}$  and  $F_{C_y}$ , as shown.

(1) *Calculation of support reactions.* Relative dimensions are shown by a small triangle on the member  $BC$  in Fig. b. From the similarity of force and relative-dimension triangles,

$$F_{C_x} = \frac{3}{5}F_C, \quad F_{C_y} = \frac{4}{5}F_C. \quad (*)$$

It follows then that  $F_{C_x} = \frac{4}{3}F_{C_y}$ . Application of equilibrium conditions to the free-body diagram in Fig. b leads to

$$\sum M_c = 0: \quad P(1.5) - F_A(2) = 0; \quad F_A = \frac{3}{4}P = 30\text{kN (right directed),}$$

$$\sum F_y = 0: \quad F_{C_y} - P = 0; \quad F_{C_y} = P = 40\text{kN (up directed),}$$

$$\sum F_x = 0: \quad -F_{C_x} + F_A = 0; \quad F_{C_x} = F_A = 30\text{kN (left directed).}$$

We thus have

$$F_C = \frac{5}{4}P = 50\text{kN}.$$

**Note.** The positive sign of  $F_A$  and  $F_C$  means that the sense of each of the forces was assumed correctly in the free-body diagram.

(2) *Calculation of internal forces.* If imaginary cutting planes are passed perpendicular to the axes of the members  $AB$  and  $BC$ , separating each into two parts, it is observed that each portion is a two-force member. Therefore the internal forces in each member are the axial forces  $F_A = 30$  kN and  $F_C = 50$  kN.

(3) *Calculation of stresses.* The normal stresses in each member are

$$\sigma_{AB} = -\frac{F_A}{A_{AB}} = -\frac{30 \times 10^3}{0.004} = -7.5\text{MPa},$$

$$\sigma_{BC} = \frac{F_C}{A_{BC}} = \frac{50 \times 10^3}{0.002} = 25\text{MPa},$$

where the minus sign indicates compression. Referring to Fig. c, we see that the double shear in the pin  $C$  is

$$\tau_c = \frac{\frac{1}{2}F_c}{\pi d^2 / 4} = \frac{25 \times 10^3}{\pi(0.02)^2 / 4} = 79.6\text{MPa}.$$

For the bearing stress in the bracket at joint  $C$ , we have

$$\sigma_b = \frac{F_c}{t_1 d} = \frac{50 \times 10^3}{(0.015)(0.02)} = 166.7 \text{ MPa},$$

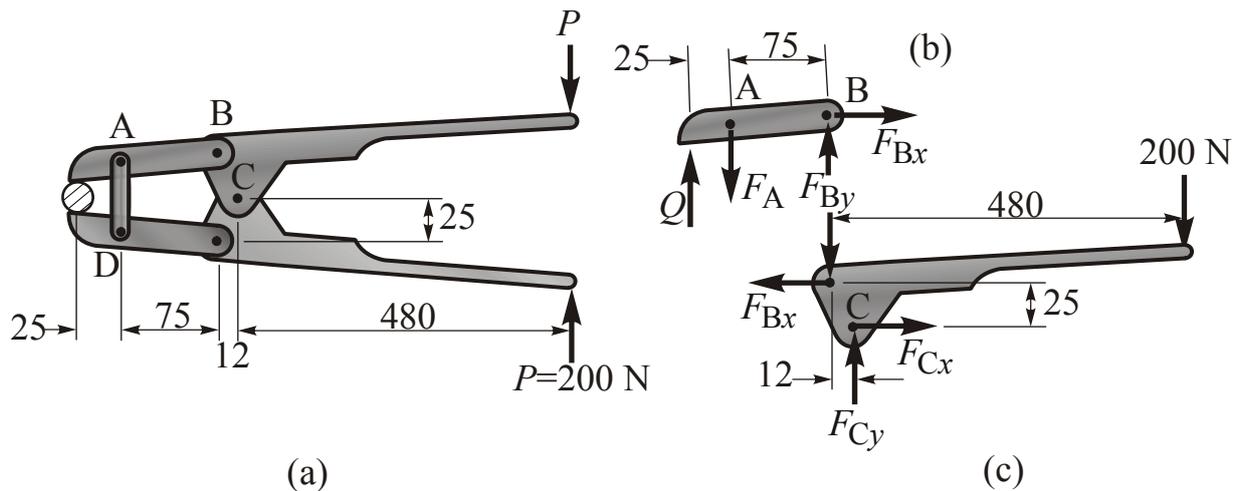
while the bearing stress in the clevis at joint  $C$  is given by

$$\sigma_b = \frac{F_c}{2td} = \frac{50 \times 10^3}{2(0.01)(0.02)} = 125 \text{ MPa}.$$

The shear and bearing stresses in the other joints may be determined in a like manner.

### Example 1.2

A force  $P$  of magnitude 200 N is applied to the handles of the bolt cutter shown in Fig. a. Compute (1) the force exerted on the bolt and rivets at joints  $A$ ,  $B$ , and  $C$  and (2) the normal stress in member  $AD$ , which has a uniform cross-sectional area of  $2 \times 10^{-4} \text{ m}^2$ . Dimensions are given in millimeters.



**Solution** The conditions of equilibrium must be satisfied by the entire cutter. To determine the unknown forces, we consider component parts. Let the force between the bolt and the jaw be  $Q$ . The free-body diagrams for the jaw and the handle are shown in Figs. b and c. Since  $AD$  is a two-force member, the orientation of force  $F_A$  is known. **Note, that the force components on the two members at joint  $B$  must be equal and opposite, as indicated in the diagrams.**

(1) Referring to the free-body diagram in Fig. b, we have

$$\sum F_x = 0: \quad F_{Bx} = 0,$$

$$\sum F_y = 0: \quad Q - F_A + F_{By} = 0, \quad F_A = Q + F_{By},$$

$$\sum M_B = 0: \quad Q(0.1) - F_A(0.075) = 0, \quad F_A = \frac{Q}{0.75},$$

from which  $Q = 3F_{By}$ . Using the free-body diagram in Fig. c, we obtain

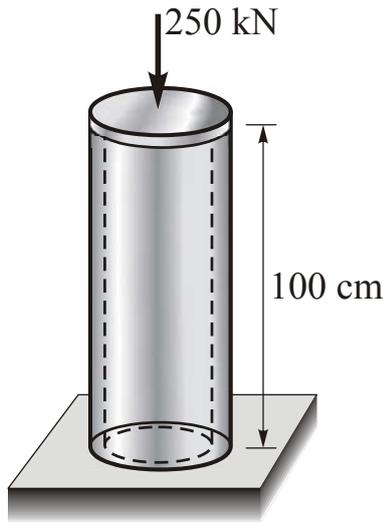
$$\begin{aligned}\sum F_x = 0: & -F_{Bx} + F_{Cx} = 0, & F_{Cx} &= 0, \\ \sum F_y = 0: & -F_{By} + F_{Cy} - 0.2 = 0, & F_{Cy} &= \frac{Q}{3} + 0.2, \\ \sum M_C = 0: & F_{Bx}(0.025) - F_{By}(0.012) + 0.2(0.48), & F_{Bx} &= 8\text{kN}.\end{aligned}$$

It follows that  $Q = 3(8) = 24$  kN. Therefore the shear forces on the rivet at the joints  $A$ ,  $B$ , and  $C$  are  $F_A = 32$  kN,  $F_B = F_{By} = 8$  kN, and  $F_C = F_{Cy} = 8.2$  kN, respectively.

(2) The normal stress in the member  $AD$  is given by

$$\sigma = \frac{F_A}{A} = \frac{32 \times 10^3}{2 \times 10^{-4}} = 160 \text{ MPa}.$$

The shear stress in the pins of the cutter is investigated as described in Example 1.1. **Note that the handles and jaws are subject to combined flexural and shearing stresses.**



Hollow aluminum post in compression

### Example 1.3

A short post constructed from a hollow circular tube of aluminum supports a compressive load of 250 kN (see figure). The inner and outer diameters of the tube are  $d_1 = 9$  cm and  $d_2 = 13$  cm, respectively, and its length is 100 cm. The shortening of the post due to the load is measured as 0.5 mm. Determine the compressive stress and strain in the post. (*Disregard the weight of the post itself, and assume that the post does not buckle under the load.*)

**Solution** Assuming that the compressive load acts at the center of the hollow tube, we can use the equation  $\sigma = P/A$  to calculate the normal stress. The force  $P$  equals 250 kN, and the cross-sectional area  $A$  is

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = \frac{\pi}{4}[(13 \text{ cm})^2 - (9 \text{ cm})^2] = 69.08 \text{ cm}^2.$$

Therefore, the compressive stress in the post is

$$\sigma = \frac{P}{A} = \frac{-250 \times 10^3 \text{ N}}{69.08 \times 10^{-4} \text{ m}^2} = -36.19 \text{ MPa}.$$

The compressive strain is

$$\varepsilon = \frac{\delta}{L} = \frac{0.5 \times 10^{-3}}{100 \times 10^{-2}} = 0.5 \times 10^{-3}.$$

**Example 1.4**

A circular steel rod of length  $L$  and diameter  $d$  hangs in a mine shaft and holds an ore bucket of weight  $W$  at its lower end (see figure). (1) Obtain a formula for the maximum stress  $\sigma_{\max}$  in the rod, taking into account the weight of the rod itself. (2) Calculate the maximum stress if  $L = 40$  m,  $d = 8$  mm, and  $W = 1.5$  kN.

**Solution** (1) The maximum axial force  $F_{\max}$  in the rod occurs at the upper end and is equal to the weight  $W$  plus the weight  $W_0$  of the rod itself. The latter is equal to the weight density  $\gamma$  of the steel times the volume  $V$  of the rod, or

$$W_0 = \gamma V = \gamma AL,$$

in which  $A$  is the cross-sectional area of the rod. Therefore, the formula for the maximum stress becomes

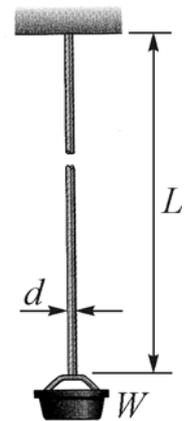
$$\sigma_{\max} = \frac{F_{\max}}{A} = \frac{W + \gamma AL}{A} = \frac{W}{A} + \gamma L.$$

(2) To calculate the maximum stress, we substitute numerical values into the preceding equation. The cross-sectional area  $A$  equals  $\frac{\pi d^2}{4}$ , where  $d = 8$  mm, and the weight

density  $\gamma$  of steel is  $77.0$  kN/m<sup>3</sup>. Thus,

$$\begin{aligned} \sigma_{\max} &= \frac{1.5 \text{ kN}}{\pi(8 \text{ mm})^2 / 4} + (77.0 \text{ kN/m}^3)(40 \text{ m}) = \\ &= 29.84 \text{ MPa} + 3.11 \text{ MPa} = 33.0 \text{ MPa}. \end{aligned}$$

**Note, the weight of the rod contributes noticeably to the maximum stress and should not be disregarded.**



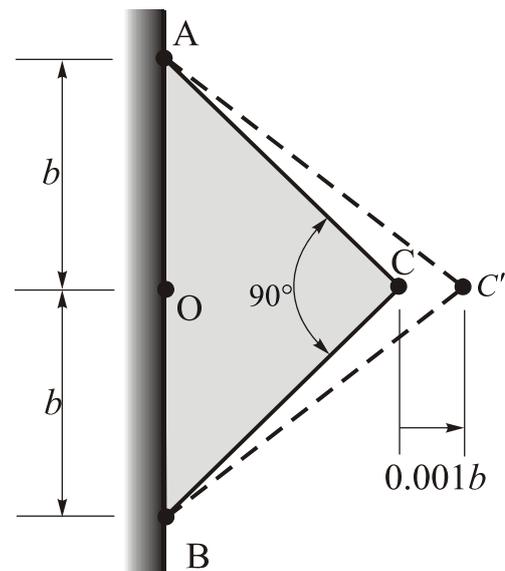
Steel rod supporting a weight  $W$

**Example 1.5**

A thin, triangular plate  $ABC$  is uniformly deformed into a shape  $ABC'$ , as shown by the dashed lines in the figure. Calculate (1) the normal strain along the centerline  $OC$ ; (2) the normal strain along the edge  $AC$ ; and (3) the shearing strain between the edges  $AC$  and  $BC$ .

**Solution** Referring to the figure, we have  $L_{OC} = b$  and

$$L_{AC} = L_{BC} = b\sqrt{2} = 1.41421b.$$



(1), (2) *Normal strains.* As the change in length  $OC$  is  $\Delta b = 0.001b$ , Eq. (1.5) yields

$$\epsilon_{oc} = \frac{0.001b}{b} = 0.001 = 1.0 \times 10^{-3}.$$

The lengths of the deformed edges are  $L_{AC'} = L_{BC'} = [b^2 + (1.001b)^2]^{1/2} = 1.41492b$ .

Thus

$$\epsilon_{AC} = \epsilon_{BC} = \frac{1.41492 - 1.41421}{1.41421} = 0.502 \times 10^{-3}.$$

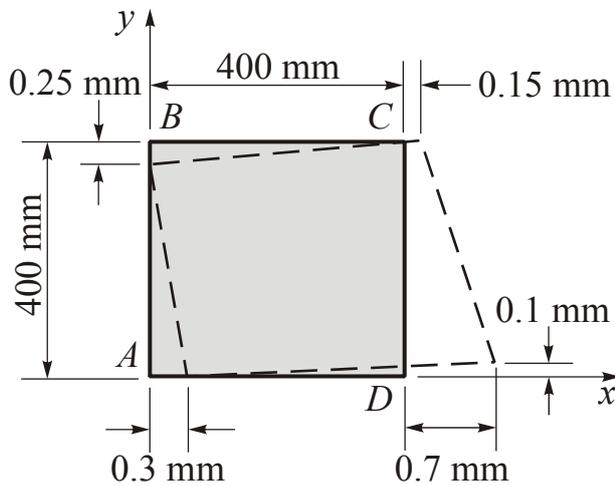
(3) *Shearing strain.* Subsequent to deformation, angle  $ACB$  becomes

$$AC'B = 2 \tan^{-1} \left( \frac{b}{1.001b} \right) = 89.943^\circ.$$

The change in the right angle is then  $90 - 89.943 = 0.057^\circ$ . The corresponding shearing strain (in radians) is

$$\gamma = 0.057 \left( \frac{\pi}{180} \right) = 0.995 \times 10^{-3}.$$

**Note.** Since the angle  $ACB$  is decreased, the shear strain is positive.



**Example 1.6**

A 0.4-m by 0.4-m square  $ABCD$  is drawn on a thin plate prior to loading. Subsequent to loading, the square has the dimensions shown by the dashed lines in the figure. Determine the average values of the plane-strain components at corner  $A$ .

**Solution** Let the original lengths of a rectangular element of unit thickness be  $\Delta x$  and  $\Delta y$ . An approximate version of Eqs. (1.8), representing

Eqs. (1.5) and (1.6), is then

$$\epsilon_x = \frac{\Delta u}{\Delta x}, \quad \epsilon_y = \frac{\Delta v}{\Delta y}, \quad \gamma_{xy} = \frac{\Delta u}{\Delta y} + \frac{\Delta v}{\Delta x}, \quad (*)$$

where  $u$  and  $v$  are, respectively, the  $x$ - and  $y$ -directed displacements of a point. For the square under consideration, we have  $\Delta x = \Delta y = 400$  mm Application of

Eqs. (\*) to the figure yields

$$\varepsilon_x = \frac{u_D - u_A}{\Delta x} = \frac{0.7 - 0.3}{400} = 10^{-3},$$

$$\varepsilon_y = \frac{v_B - v_A}{\Delta y} = \frac{-0.25 - 0}{400} = -0.625 \times 10^{-3}.$$

Similarly,

$$\gamma_{xy} = \frac{u_B - u_A}{\Delta y} + \frac{v_D - v_A}{\Delta x} = \frac{0 - 0.3}{400} + \frac{0.1 - 0}{400} = -0.5 \times 10^{-3}.$$

**Note.** The negative sign indicates that angle  $BAD$  has increased.

### Example 1.7

A steel strut  $S$  serving as a brace for a boat hoist transmits a compressive force  $P = 54 \text{ kN}$  to the deck of a pier (see figure (a)). The strut has a hollow square cross section with a wall thickness  $t = 12 \text{ mm}$  (see figure (b)), and the angle  $\theta$  between the strut and the horizontal is  $40^\circ$ . A pin through the strut transmits the compressive force from the strut to two gussets  $G$  that are welded to the base plate  $B$ . Four anchor bolts fasten the base plate to the deck. The diameter of the pin is  $d_{pin} = 18 \text{ mm}$ , the thickness of the gussets is  $t_G = 15 \text{ mm}$ , the thickness of the base plate is  $t_B = 8 \text{ mm}$ , and the diameter of the anchor bolts is  $d_{bolt} = 12 \text{ mm}$ .

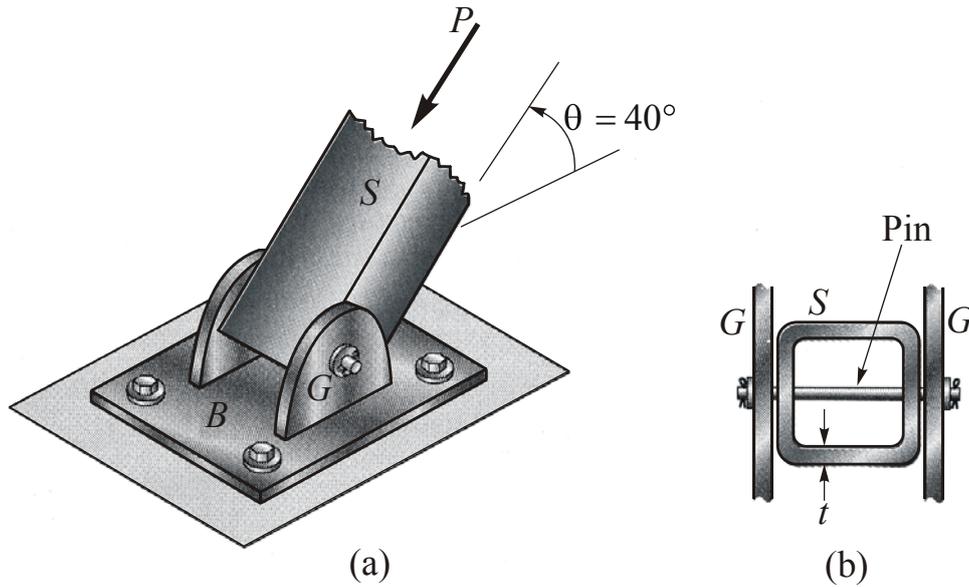
Determine the following stresses: (1) the bearing stress between the strut and the pin; (2) the shear stress in the pin; (3) the bearing stress between the pin and the gussets; (4) the bearing stress between the anchor bolts and the base plate, and (5) the shear stress in the anchor bolts. In solution, disregard any friction between the base plate and the deck.

**Solution** (1) *Bearing stress between strut and pin.* The average value of the bearing stress between the strut and the pin is found by dividing the force in the strut by the total bearing area of the strut against the pin. The latter is equal to twice the thickness of the strut (because bearing occurs at two locations) times the diameter of the pin (see figure (b)). Thus, the bearing stress is

$$\sigma_{b1} = \frac{P}{2td_{pin}} = \frac{54 \text{ kN}}{2(12 \text{ mm})(18 \text{ mm})} = 125 \text{ MPa}.$$

This, stress is not excessive for a strut made of structural steel, since the yield stress is probably near  $200 \text{ MPa}$  (see Appendix A). Assuming the factor of safety  $f_s = 1.5$  allowable stress  $\sigma_{allow} = 133 \text{ MPa}$ . It means that the strut will be strong in bearing.

(2) *Shear stress in pin.* As can be seen from figure (b), the pin tends to shear on two planes, namely, the planes between the strut and the gussets.



(a) Pin connection between strut S and base plate B. (b) Cross section through the strut S

Therefore, the average shear stress in the pin (which is in double shear) is equal to the total load applied to the pin divided by twice its cross-sectional area:

$$\tau_{\text{pin}} = \frac{P}{2\pi d_{\text{pin}}^2 / 4} = \frac{54 \text{ kN}}{2\pi(18 \text{ mm})^2 / 4} = 106 \text{ MPa}.$$

The pin would normally be made of high-strength steel (tensile yield stress greater than 340 MPa) and could easily withstand this shear stress (*the yield stress in shear is usually at least 50% of the yield stress in tension*).

(3) *Bearing stress between pin and gussets.* The pin bears against the gussets at two locations, so the bearing area is twice the thickness of the gussets times the pin diameter; thus,

$$\sigma_{b2} = \frac{P}{2t_G d_{\text{pin}}} = \frac{54 \text{ kN}}{2(15 \text{ mm})(18 \text{ mm})} = 100 \text{ MPa},$$

which is less than the bearing stress against the strut.

(4) *Bearing stress between anchor bolts and base plate.* The vertical component of the force P (see figure (a)) is transmitted to the pier by direct bearing between the base plate and the pier. The horizontal component, however, is transmitted through the anchor bolts. The average bearing stress between the base plate and the anchor bolts is equal to the horizontal component of the force P divided by the bearing area of four bolts. The bearing area for one bolt is equal to the thickness of the plate times the bolt diameter. Consequently, the bearing stress is

$$\sigma_{b3} = \frac{P \cos 40^\circ}{4t_B d_{\text{bolt}}} = \frac{(54 \text{ kN})(\cos 40^\circ)}{4(8 \text{ mm})(12 \text{ mm})} = 108 \text{ MPa}.$$

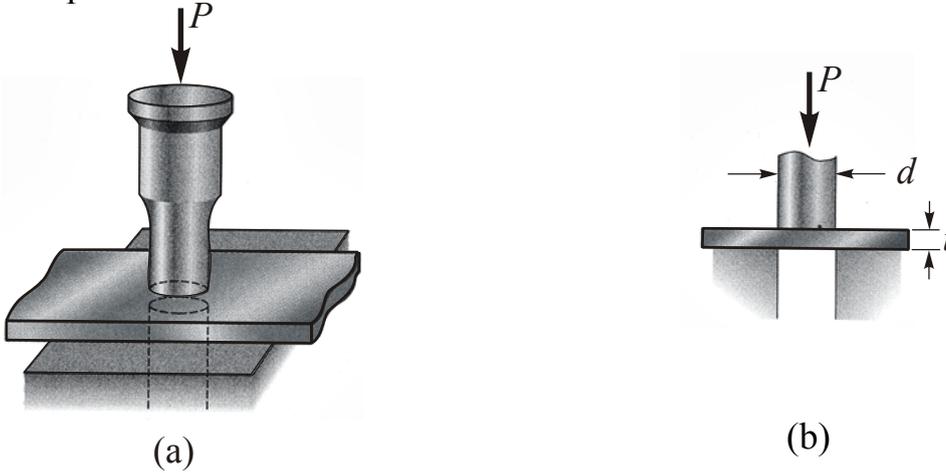
(5) *Shear stress in anchor bolts.* The average shear stress in the anchor bolts is equal to the horizontal component of the force  $P$  divided by the total cross-sectional area of four bolts (note that each bolt is in single shear). Therefore,

$$\tau_{\text{bolt}} = \frac{P \cos 40^\circ}{4\pi d_{\text{bolt}}^2 / 4} = \frac{(54 \text{ kN})(\cos 40^\circ)}{4\pi(12 \text{ mm})^2 / 4} = 119 \text{ MPa} .$$

**Note.** Any friction between the base plate and the pier would reduce the load on the anchor bolts.

### Example 1.8

A punch for making holes in steel plates is shown in Fig. a. Assume that a punch having a diameter of 19 mm is used to punch a hole in a 6-mm plate, as shown in the cross-sectional view (see figure (b)). If a force  $P=125 \text{ kN}$  is required, what is the average shear stress in the plate and the average compressive stress in the punch?



Punching a hole in a steel plate

**Solution** The average shear stress in the plate is obtained by dividing the force  $P$  by the shear area of the plate. The shear area as is equal to the circumference of the hole times the thickness of the plate, or

$$A_s = \pi dt = \pi(19 \text{ mm})(6 \text{ mm}) = 358 \text{ mm}^2 ,$$

in which  $d$  is the diameter of the punch and  $t$  is the thickness of the plate. Therefore, the average shear stress in the plate is

$$\tau_{\text{aver}} = \frac{P}{A_s} = \frac{125,000 \text{ N}}{358 \times 10^{-6} \text{ m}^2} = 349 \text{ MPa} .$$

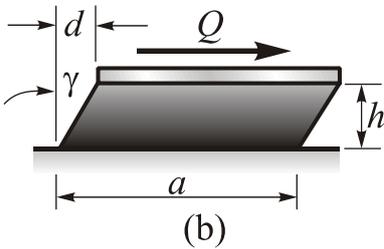
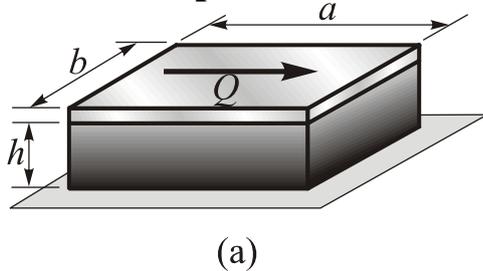
The average compressive stress in the punch is

$$\sigma_c = \frac{P}{A_{\text{punch}}} = \frac{P}{\pi d^2 / 4} = \frac{125,000 \text{ N}}{\pi(19 \times 10^{-3})^2 / 4} = 441 \text{ MPa} ,$$

in which  $A_{\text{punch}}$  is the cross-sectional area of the punch.

**Note.** This analysis is highly idealized because we are disregarding impact effects that occur when a punch is rammed through a plate.

**Example 1.9**



Bearing pad in shear

A bearing pad of the kind used to support machines and bridge girders consists of a linearly elastic material (usually an elastomer, such as rubber) capped by a steel plate (see figure (a)). Assume that the thickness of the elastomer is  $h$ , the dimensions of the plate are  $a \times b$ , and the pad is subjected to a horizontal shear force  $Q$ .

Obtain formulas for the average shear stress  $\tau_{aver}$  in the elastomer and the horizontal displacement  $d$  of the plate (see figure (b)).

**Solution** Assume that the shear stresses in the elastomer are uniformly distributed throughout its entire volume. Then the shear stress on any horizontal plane through the elastomer equals the shear force  $Q$  divided by the area of the plane (see figure (a)):

$$\tau_{aver} = \frac{Q}{ab}.$$

The corresponding shear strain (from Hooke's law in shear, which will be considered below) is

$$\gamma = \frac{\tau_{aver}}{G_e} = \frac{Q}{abG_e},$$

in which  $G_e$  is the shear modulus of the elastomeric material. Finally, the horizontal displacement  $d$  is equal to  $h \tan \gamma$ :

$$d = h \tan \gamma = h \tan \left( \frac{Q}{abG_e} \right).$$

In most practical situations the shear strain  $\gamma$  is a small angle, and in such cases we may replace  $\tan \gamma$  by  $\gamma$  and obtain

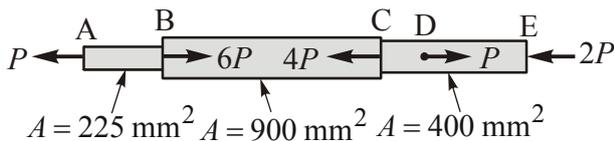
$$d = h\gamma = \frac{hQ}{abG_e}.$$

Equations mentioned above give approximate results for the horizontal displacement of the plate because they are based upon the assumption that the

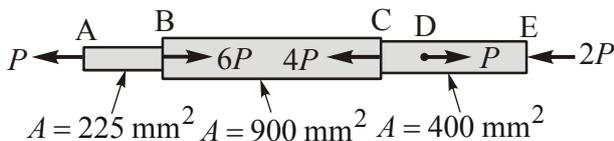
shear stress and strain are constant throughout the volume of the elastomeric material. In reality the shear stress is zero at the edges of the material (because there are no shear stresses on the free vertical faces), and therefore the deformation of the material is more complex than pictured in the figure. However, if the length  $a$  of the plate is large compared with the thickness  $h$  of the elastomer, the results are satisfactory for design purposes.

 **PROBLEMS**

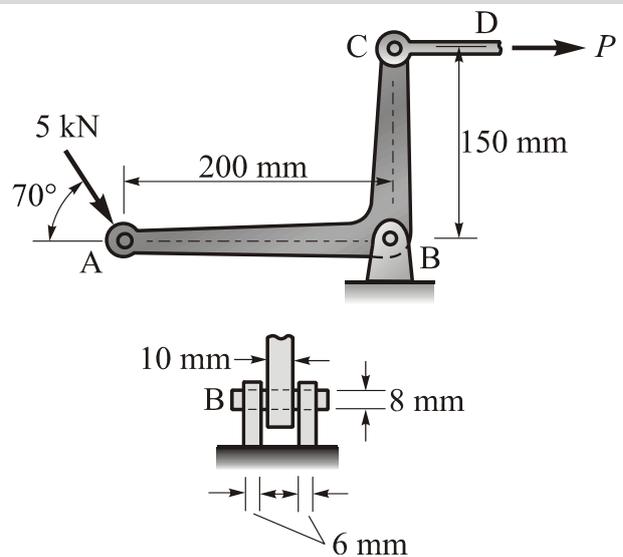
**Problem 1.1** Determine the normal stress in each segment of the stepped bar shown in the figure. Load  $P = 20$  kN.



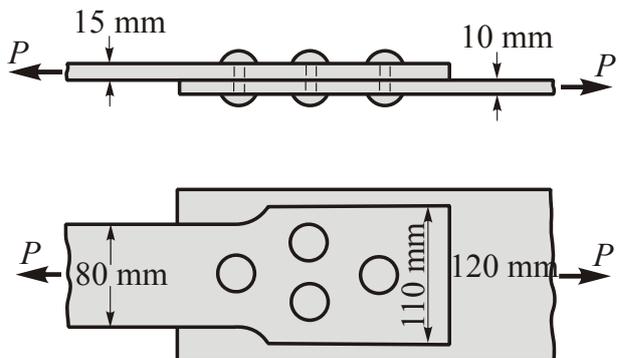
**Problem 1.2** A rod is subjected to the five forces shown in the figure. What is the maximum value of  $P$  for the stresses not to exceed 100 MPa in tension and 140 MPa in compression?



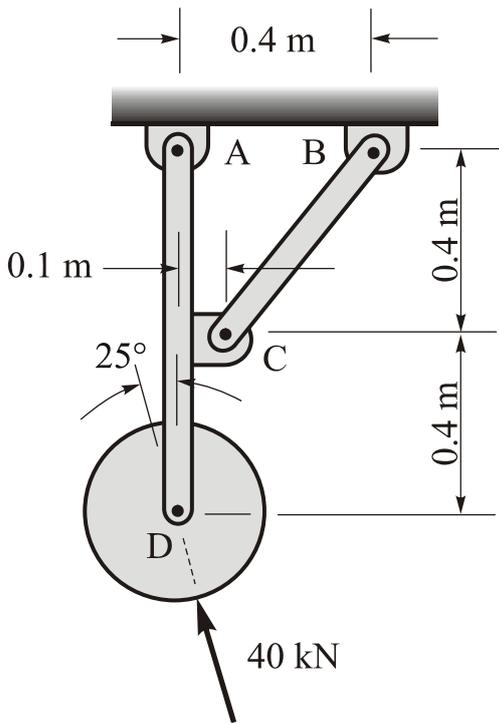
**Problem 1.3** The bell-crank mechanism shown in the figure is in equilibrium. Determine (1) the normal stress in the connecting rod  $CD$ ; (2) the shearing stress in the 8-mm-diameter pin at  $B$  point; (3) the bearing stress in the bracket supports at  $B$ ; and (4) the bearing stress in the crank at  $B$  point.



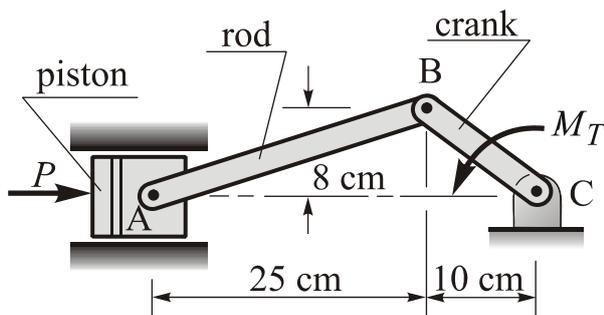
**Problem 1.4** Two plates are joined by four rivets of 20-mm diameter, as shown in the figure. Determine the maximum load  $P$  if the shearing, tensile, and bearing stresses are limited to 80, 100, and 140 MPa, respectively. Assume that the load is equally divided among the rivets.



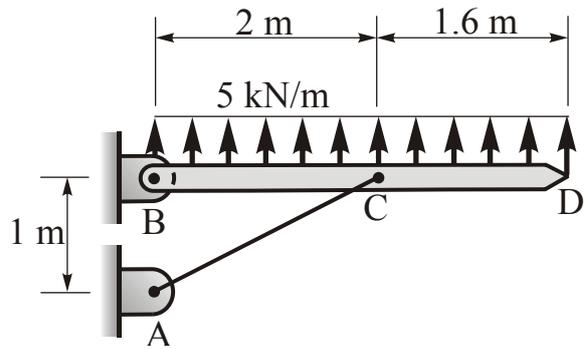
**Problem 1.5** Calculate the shearing stresses produced in the pins at *A* and *B* for the landing gear shown in the figure. Assume that each pin has a diameter of 25 mm and is in double shear.



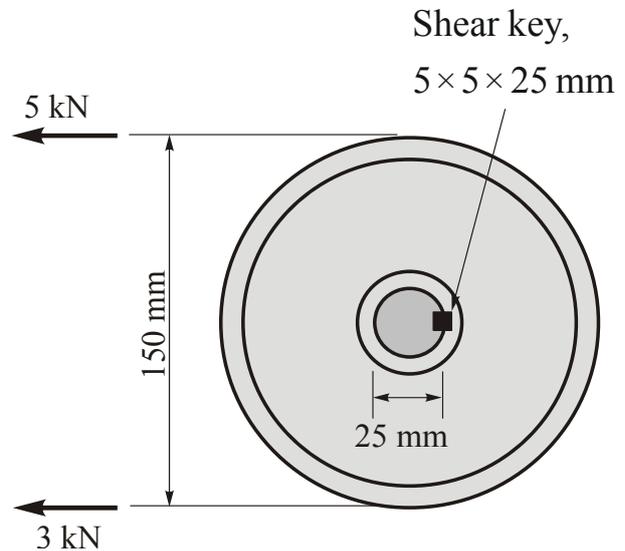
**Problem 1.6** The piston, connecting rod, and crank of an engine system are depicted in the figure. Assuming that a force  $P = 10$  kN acts as indicated, determine (1) the torque  $M_T$  required to hold the system in equilibrium and (2) the normal stress in the rod *AB* if its cross-sectional area is  $5 \text{ cm}^2$ .



**Problem 1.7** The wing of a monoplane is shown in the figure. Determine the normal stress in rod *AC* of the wing if it has a uniform cross section of  $20 \times 10^{-5} \text{ m}^2$ .

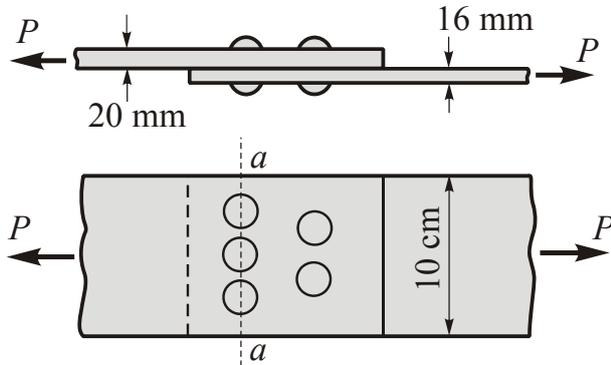


**Problem 1.8** A 150-mm pulley subjected to the loads shown in the figure is keyed to a shaft of 25-mm diameter. Calculate the shear stress in the key.

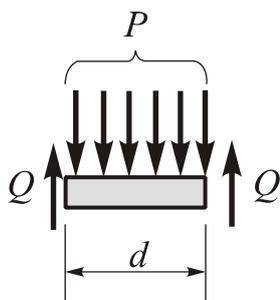
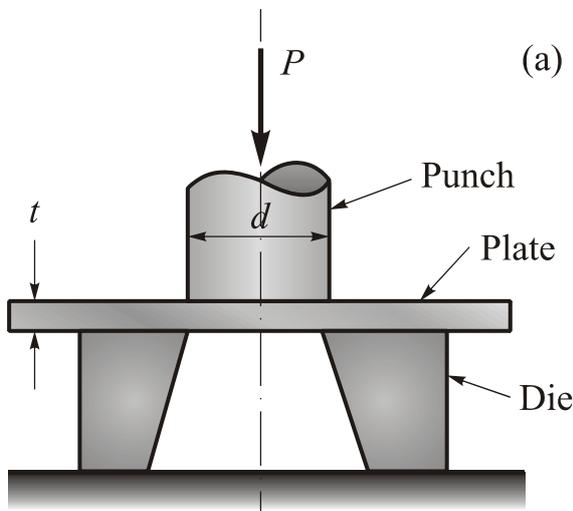


**Problem 1.9** The lap joint seen in the figure is fastened by five 2.5-cm-diameter rivets. For  $P = 50$  kN, determine (1) the maximum shear stress in the rivets; (2) the maximum bearing stress; and (3) the maximum tensile stress at section *a-a*. Assume that the

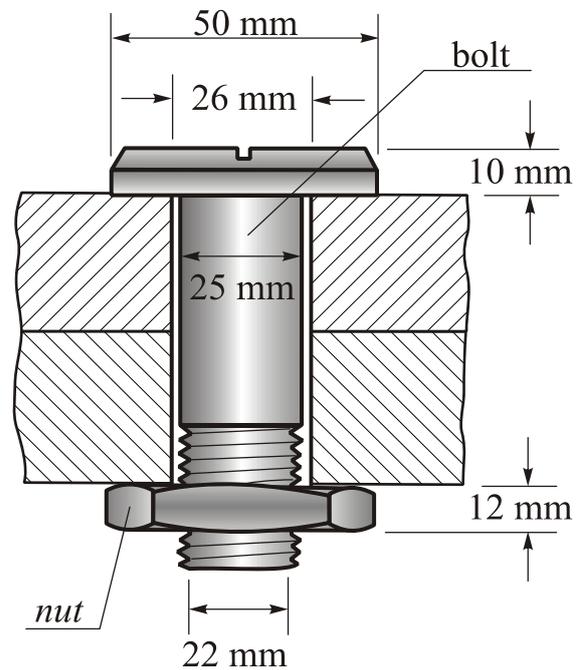
load is divided equally among the rivets.



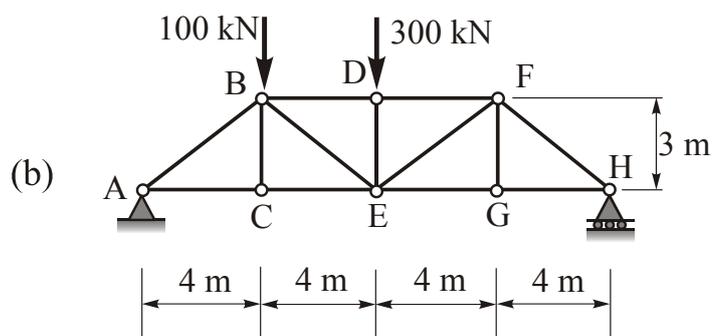
**Problem 1.10** A punch having a diameter  $d$  of 2.5 cm is used to punch a hole in a steel plate with a thickness  $t$  of 10 mm, so illustrated in the figure. Calculate: (1) the force  $P$  required if the shear stress in steel is 140 MPa and (2) the corresponding normal stress in the punch.



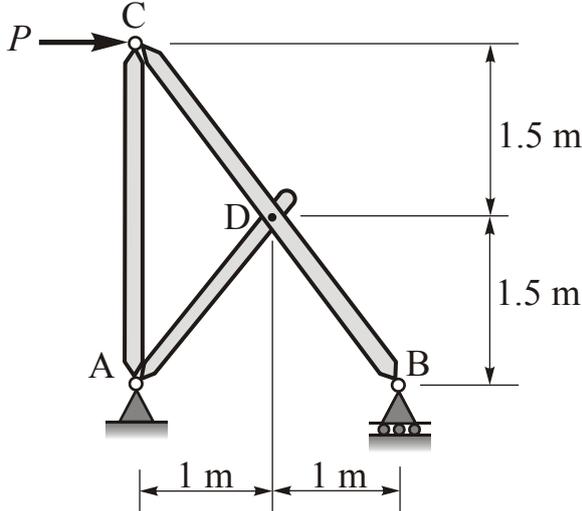
**Problem 1.11** Two plates are fastened by a bolt as shown in the figure. The nut is tightened to cause a tensile load in the shank of the bolt of 60 kN. Determine (1) the shearing stress in the threads; (2) the shearing stress in the head of the bolt; (3) the bearing stress between the head of the bolt and the plate; and (4) the normal stress in the bolt shank.



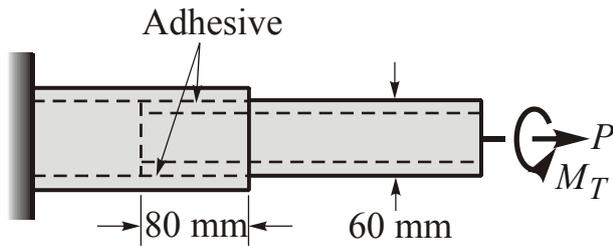
**Problem 1.12** Determine the stresses in members  $BE$  and  $CE$  of the pin-connected truss shown in the figure. Each bar has a uniform cross-sectional area of  $5 \times 10^{-3} \text{ m}^2$ .



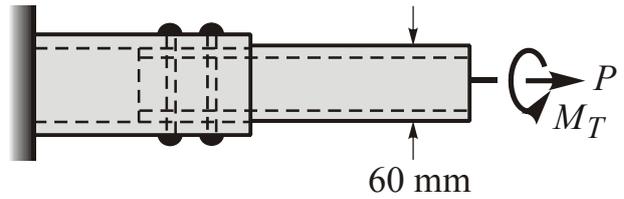
**Problem 1.13** The frame shown in the figure consists of three pin-connected, 2.5-cm-diameter bars. Calculate the normal stresses in bars  $AC$  and  $AD$  for  $P = 5$  kN.



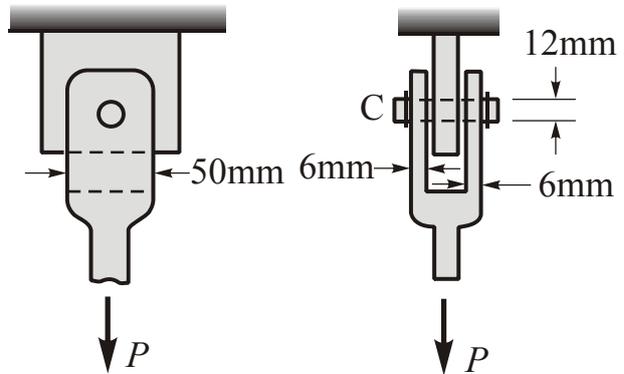
**Problem 1.14** The two tubes shown in the figure are joined with an adhesive of shear strength  $\tau = 2$  MPa. Determine (1) the maximum axial load  $P$  the joint can transmit and (2) the maximum torque moment  $M_T$  the joint can transmit.



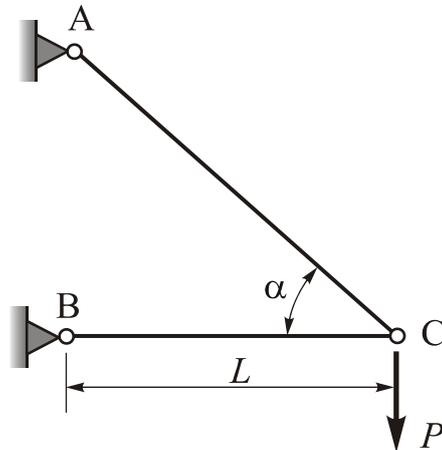
**Problem 1.15** The two tubes shown in the figure are joined by two 10-mm-diameter rivets, each having a shear strength of  $\tau = 70$  MPa. Determine (1) the maximum axial load  $P$  the joint can transmit and (2) the maximum torque moment  $M_T$  the joint can transmit.



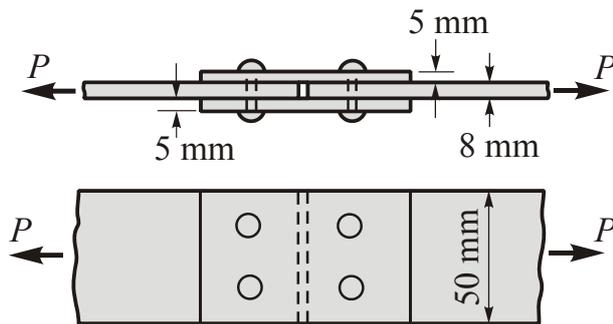
**Problem 1.16** The connection shown in the figure is subjected to a load  $P = 20$  kN. Calculate (1) the shear stress in the pin at  $C$ ; (2) the maximum tensile stress in the clevis; and (3) the bearing stress in the clevis at  $C$ .



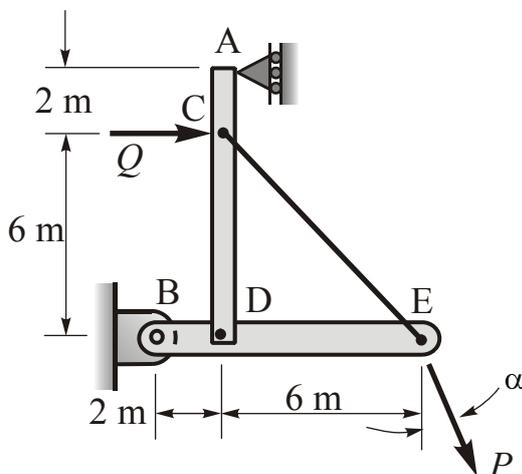
**Problem 1.17** Two rods  $AC$  and  $BC$  are connected by pins to form a mechanism for supporting a vertical load  $P$  at  $C$ , as shown in the figure. The normal stresses  $\sigma$  in both rods are to be equal. Determine the angle  $\alpha$  if the frame is to be of minimum weight.



**Problem 1.18** The butt joint (see figure) is fastened by four 15-mm-diameter rivets. Determine the maximum load  $P$  if the stresses are not to exceed 100 MPa in shear, 140 MPa in tension, and 200 MPa in bearing. Assume that the load is equally divided among the rivets.



**Problem 1.19** The pin-connected frame shown in the figure supports the loads  $Q = 5 \text{ kN}$  and  $P = 10 \text{ kN}$ . Determine, for  $\alpha = 30^\circ$ , (1) the normal stress in the bar  $CE$  of uniform cross-sectional area  $15 \times 10^{-5} \text{ m}^2$  and (2) the shearing stresses in the 10-mm-diameter pins at  $D$  and  $E$  if both are in double shear.



**Problem 1.20** A long aluminum alloy wire of weight density  $\gamma = 28 \text{ kN/m}^3$  and yield strength 280 MPa hangs vertically under its own weight. Calculate the greatest length it can have without permanent deformation.

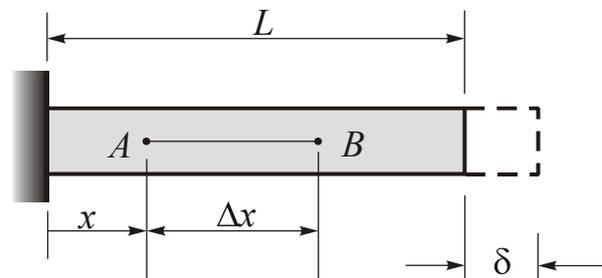
**Problem 1.21** A spherical balloon changes its diameter from 200 to 201 mm when pressurized. Determine the average circumferential strain.

**Problem 1.22** A hollow cylinder is subjected to an internal pressure which increases its 200-mm inner diameter by 0.5 mm and its 400-mm outer diameter by 0.3 mm. Calculate (1) the maximum normal strain in the circumferential direction and (2) the average normal strain in the radial direction.

**Problem 1.23** Calculate the maximum strain  $\epsilon_x$  in the bar seen in the figure if the displacement along the member varies as

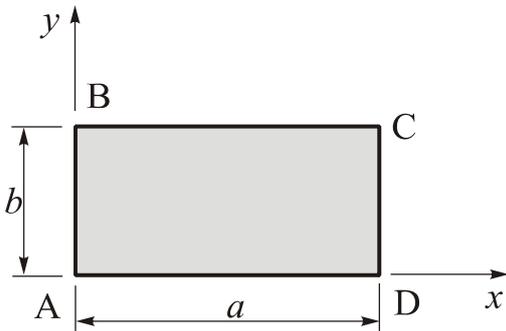
$$(1) u(x) = (x^2/L) \times 10^{-3} \text{ and}$$

$$(2) u(x) = L \times (10^{-3}) \sin(\pi x/2L).$$

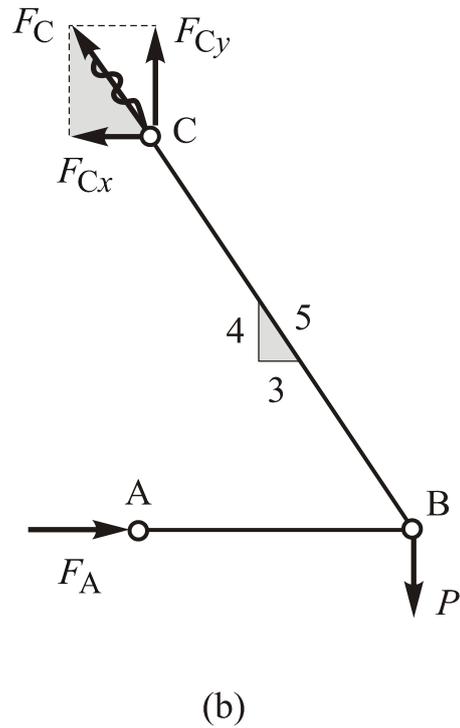
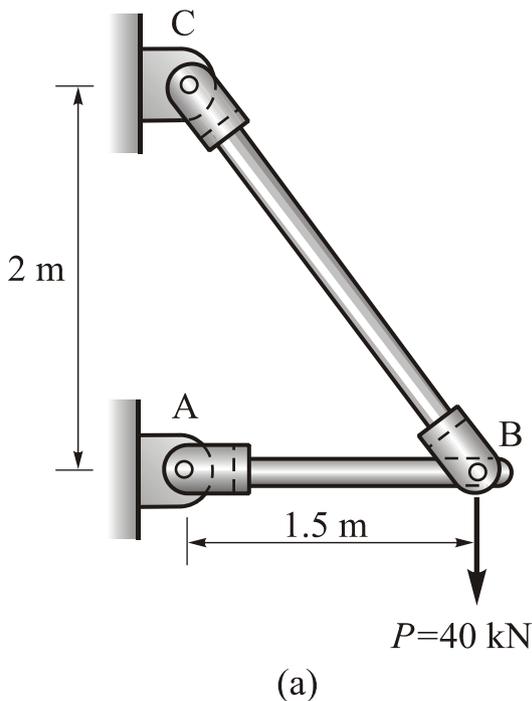


**Problem 1.24** As a result of loading, the thin rectangular plate (see the figure) deforms into a parallelogram

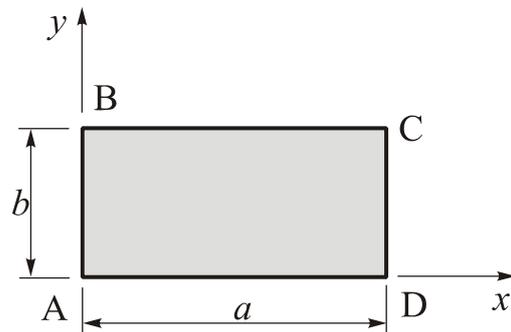
in which sides  $AB$  and  $CD$  elongate  $0.005$  mm and rotate  $1200 \times 10^{-6}$  rad clockwise, while sides  $AD$  and  $BC$  shorten  $0.002$  mm and rotate  $400 \times 10^{-6}$  rad counterclockwise. Calculate the plane strain components. Use  $a = 30$  mm and  $b = 20$  mm.



**Problem 1.25** Determine the normal strain in the members  $AB$  and  $CB$  of the structure shown in the figure if point  $B$  is displaced leftward 3 mm.



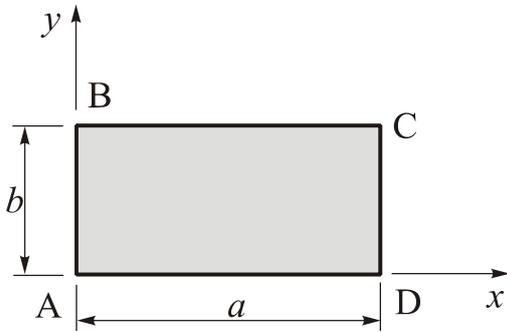
**Problem 1.26** A thin rectangular plate,  $a = 20$  cm and  $b = 10$  cm (see figure), is acted upon by a biaxial tensile loading resulting in the uniform strains  $\epsilon_x = 0.6 \times 10^{-3}$  and  $\epsilon_y = 0.4 \times 10^{-3}$ . Calculate the change in length of diagonal  $AC$ .



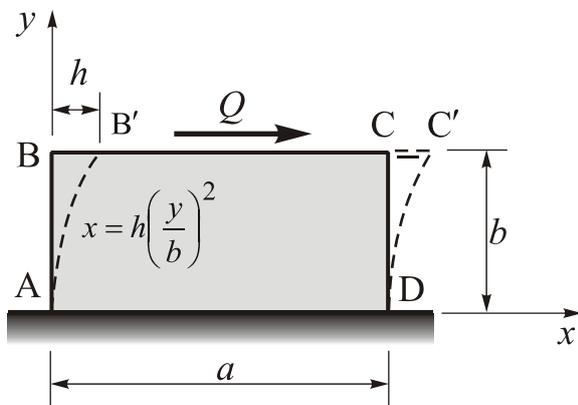
**Problem 1.27** A thin rectangular plate,  $a = 20$  cm and  $b = 10$  cm (see figure), is acted upon by a biaxial compressive loading resulting in

the uniform strains  $\epsilon_x = -0.2 \times 10^{-3}$  and  $\epsilon_y = -0.1 \times 10^{-3}$ . Calculate the change in length of diagonal  $AC$ .

$\epsilon_x = -0.5 \times 10^{-3}$ ,  $\epsilon_y = 0.5 \times 10^{-3}$  and  $\gamma_{xy} = 0$ . Calculate the negative shearing strain between its diagonals.



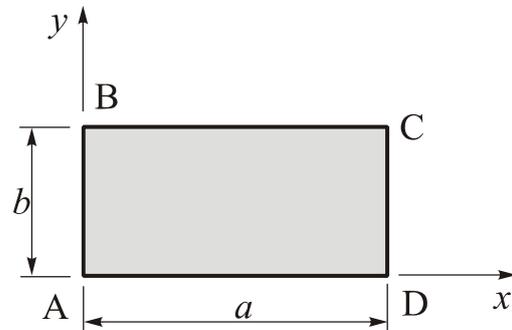
**Problem 1.28** The shear force  $Q$  deforms plate  $ABCD$  into  $AB'C'D$  (see figure). For  $b = 200$  mm and  $h = 0.5$  mm, determine the shearing strain in the plate (1) at any point; (2) at the center; and (3) at the origin.



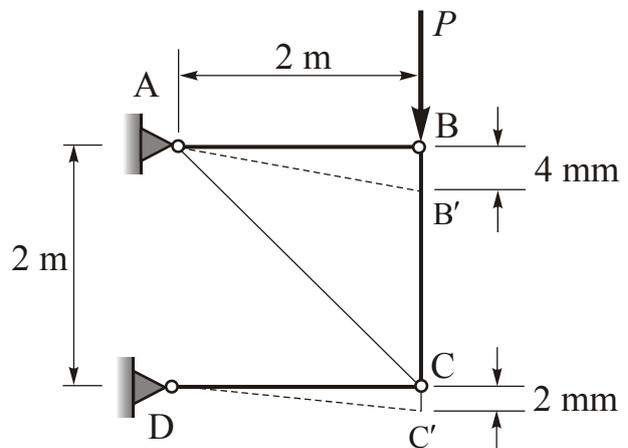
**Problem 1.29** A 100-mm by 100-mm square plate is deformed into a 100-mm by 100.2-mm rectangle. Determine the positive shear strain between its diagonals.

**Problem 1.30** A square plate is subjected to uniform strains

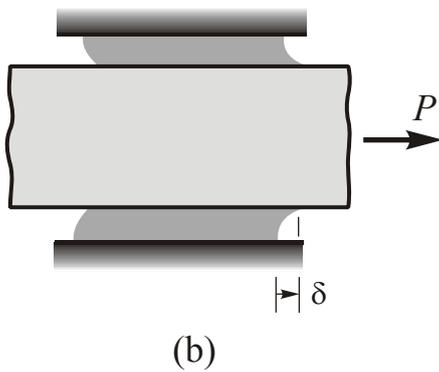
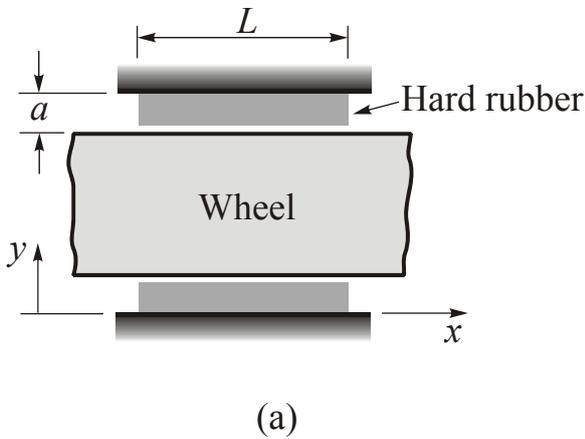
**Problem 1.31** The plate (see figure) deforms in loading into a shape in which diagonal  $BD$  elongates 0.2 mm and diagonal  $AC$  contracts 0.4 mm while they remain perpendicular and side  $AD$  remains horizontal. Calculate the average plane strain components. Take  $a = b = 400$  mm.



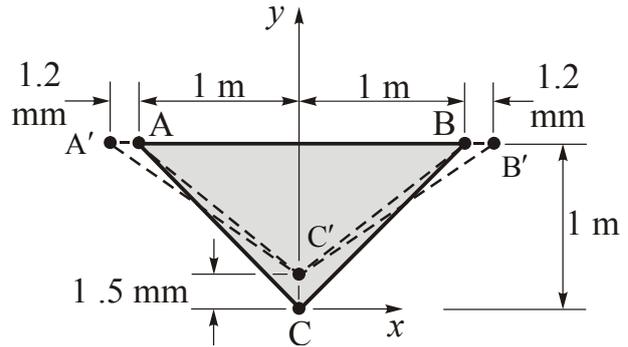
**Problem 1.32** The pin-connected structure  $ABCD$  is deformed into a shape  $AB'C'D$ , as shown by the dashed lines in the figure. Calculate the average normal strains in members  $BC$  and  $AC$ .



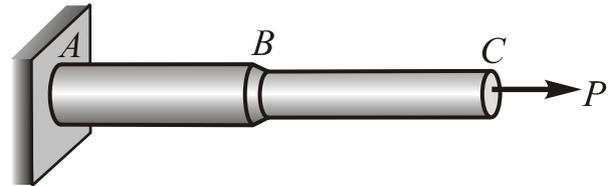
**Problem 1.33** The handbrakes on a bicycle consist of two blocks of hard rubber attached to the frame of the bike, which press against the wheel during stopping (see figure (a)). Assuming that a force  $P$  causes a parabolic deflection ( $x = ky^2$ ) of the rubber when the brakes are applied (see figure (b)), determine the shearing strain in the rubber.



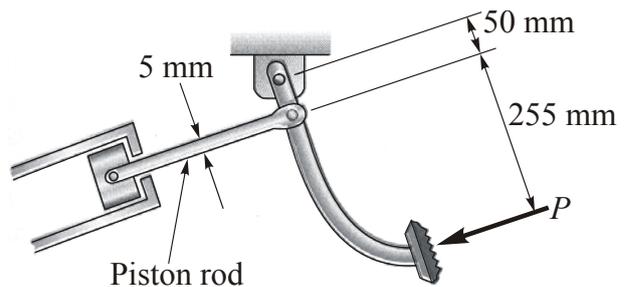
**Problem 1.34** The thin, rectangular plate  $ABC$  shown in the figure is uniformly deformed into a shape  $A'B'C'$ . Calculate: (1) the plane strain components  $\epsilon_x$ ,  $\epsilon_y$ , and  $\gamma_{xy}$  and (2) the shearing strain between edges  $AC$  and  $BC$ .



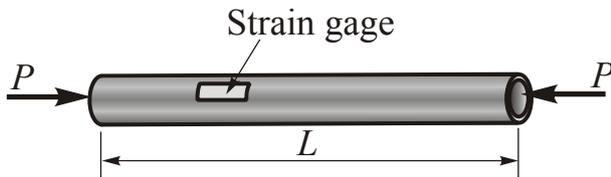
**Problem 1.35** A metal bar  $ABC$  having two different cross-sectional areas is loaded by an axial force  $P$  (see figure). Parts  $AB$  and  $BC$  are circular in cross section with diameters 45 mm and 32 mm, respectively. If the normal stress in part  $AB$  is 35 MPa, what is the normal stress  $\sigma_{BC}$  in part  $BC$ ?



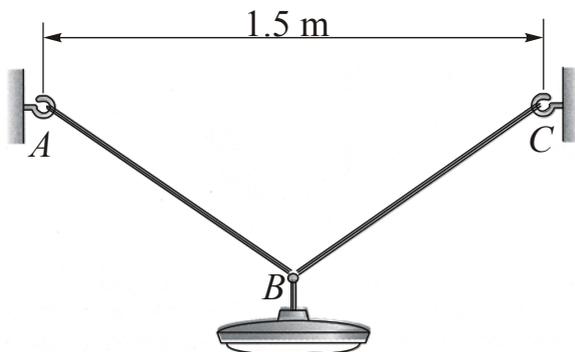
**Problem 1.36** Calculate the compressive stress  $\sigma_c$  in the piston rod (see figure) when a force  $P = 40\text{ N}$  is applied to the brake pedal. The line of action of the force  $P$  is parallel to the piston rod. Also, the diameter of the piston rod is 5 mm, and the other dimensions shown in the figure are measured perpendicular to the line of action of the force  $P$ .



**Problem 1.37** A circular aluminum tube of length  $L = 50\text{ cm}$  is loaded in compression by forces  $P$  (see figure). The outside and inside diameters are 6 cm and 5 cm, respectively. A strain gage is placed on the outside of the bar to measure normal strains in the longitudinal direction. (1) If the measured strain  $\epsilon = -570 \times 10^{-6}$ , what is the shortening  $\delta$  of the bar? (2) If the compressive stress in the bar is intended to be 40 MPa, what should be the load  $P$ ?

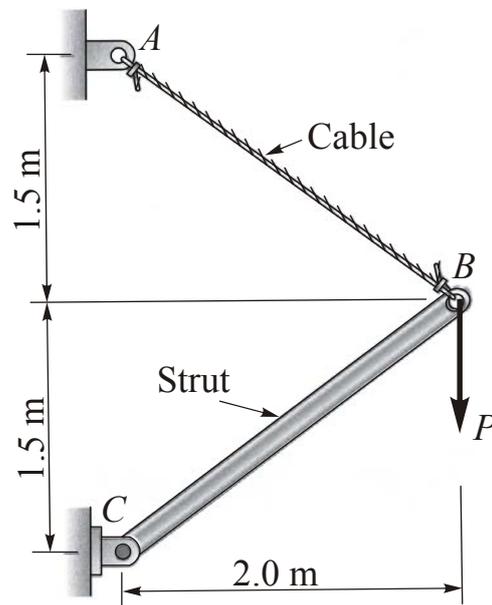


**Problem 1.38** A steel wire  $ABC$  supporting a lamp at its midpoint is attached to supports that are 1.5 m apart (see figure). The length of the wire is 2 m and its diameter is 0.5 mm. If the lamp weighs 60 N, what is the tensile stress  $\sigma_t$  in the wire?

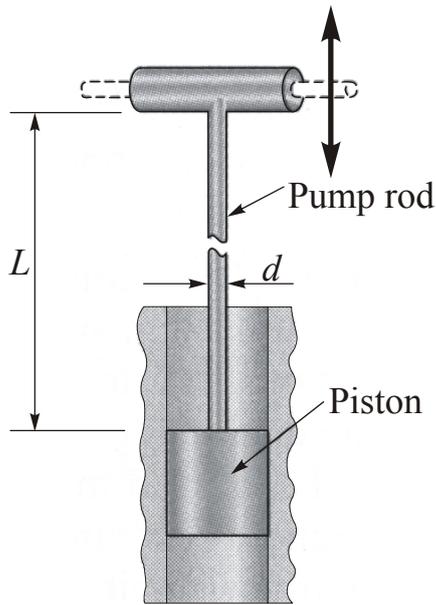


**Problem 1.39** A cable and strut assembly  $ABC$  (see figure) supports a vertical load  $P = 12\text{ kN}$ . The cable has

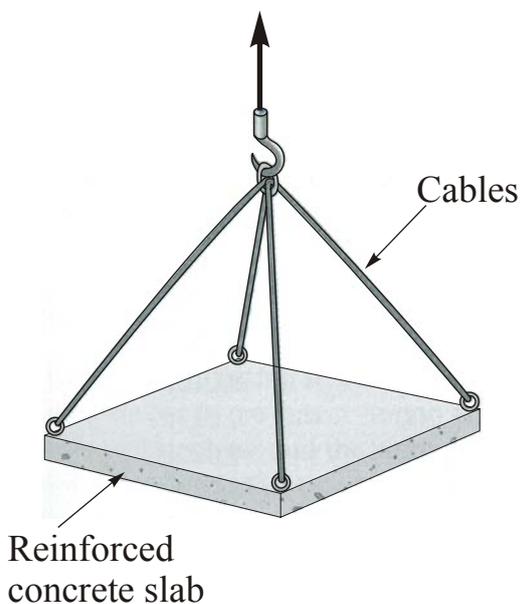
an effective cross-sectional area of  $150\text{ mm}^2$ , and the strut has an area of  $300\text{ mm}^2$ . (1) Calculate the normal stresses  $\sigma_{AB}$  and  $\sigma_{BC}$  in the cable and strut, respectively, and indicate whether they are tensiled or compressed. (2) If the cable elongates 1.1 mm, what is the strain? (3) If the strut shortens 0.35 mm, what is the strain?



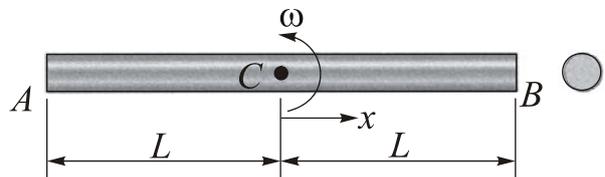
**Problem 1.40** A pump moves a piston up and down in a deep water well (see figure). The pump rod has diameter  $d = 20\text{ mm}$  and length  $L = 100\text{ cm}$ . The rod is made of steel having weight density  $\gamma = 77.0\text{ kN/m}^3$ . The resisting force associated with the piston during the downstroke is 900 N and during the upstroke is 10,800 N. Determine the maximum tensile and compressive stresses in the pump rod due to the combined effects of the resistance forces and the weight of the rod.



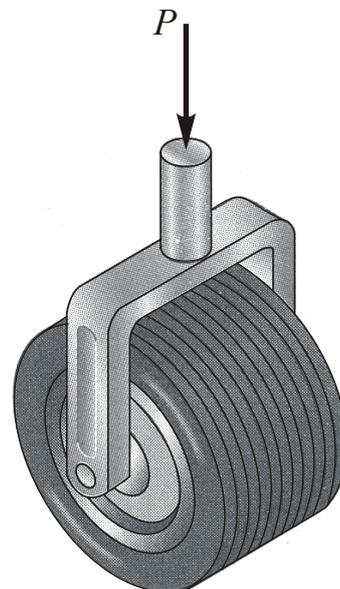
**Problem 1.41** A reinforced concrete slab 2.5 m square and 20 cm thick is lifted by four cables attached to the corners, as shown in the figure. The cables are attached to a hook at a point 1.5 m above the top of the slab. The cables have an effective cross-sectional area  $A = 80 \text{ mm}^2$ . Determine the tensile stress  $\sigma_t$  in the cables. The concrete specific weight (weight density)  $15,000 \text{ N/m}^3$ .



**Problem 1.42** A round bar  $ACB$  of total length  $2L$  (see figure) rotates about an axis through the midpoint  $C$  with constant angular speed  $\omega$  (radians per second). The material of the bar has weight density  $\gamma$ . (1) Derive a formula for the tensile stress  $\sigma_x$  in the bar as a function of the distance  $x$  from the midpoint  $C$ . (2) What is the maximum tensile stress  $\sigma_{\max}$ ?

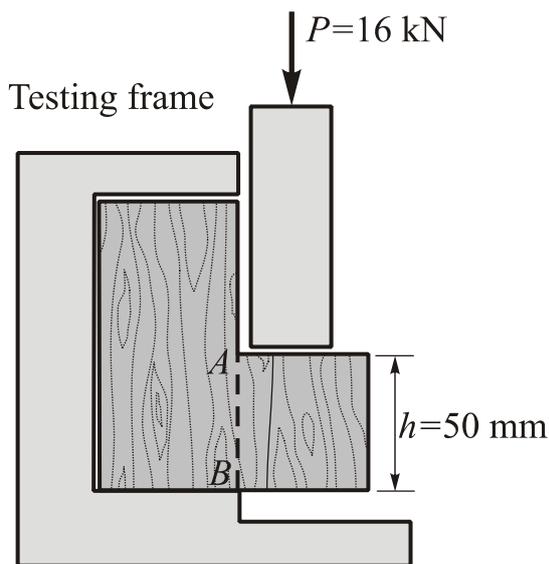


**Problem 1.43** The vertical load  $P$  acting on the wheel of a vehicle is 60 kN (see figure). What is the average shear stress  $\tau_{\text{aver}}$  in the 30-mm diameter axle?



**Problem 1.44** A block of wood is tested in direct shear using the testing frame and test specimen shown in the

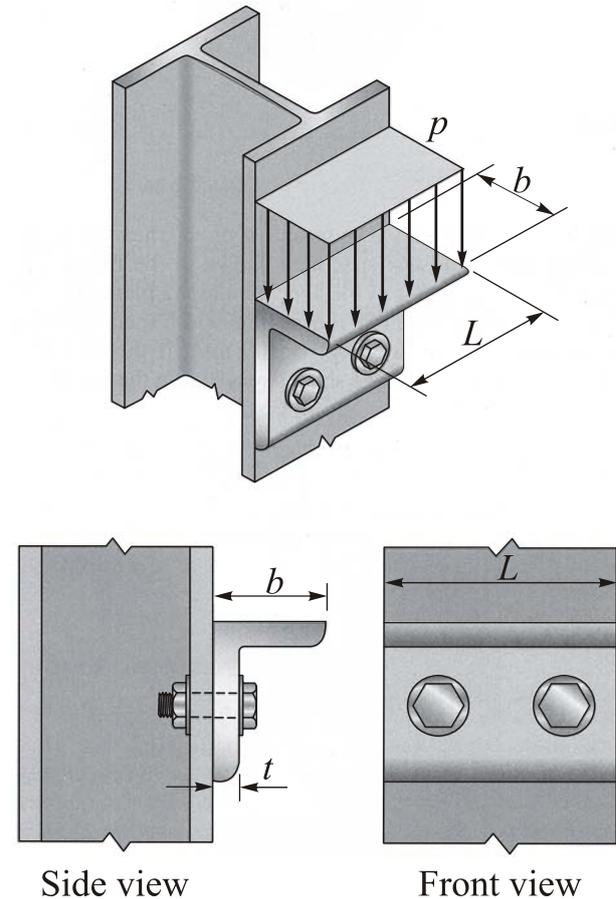
figure. The load  $P$  produces shear in the specimen along plane  $AB$ . The height  $h$  of plane  $AB$  is 50 mm and its width (perpendicular to the plane of the drawing) is 100 mm. If the load  $P=16$  kN, what is the average shear stress  $\tau_{aver}$  in the wood?



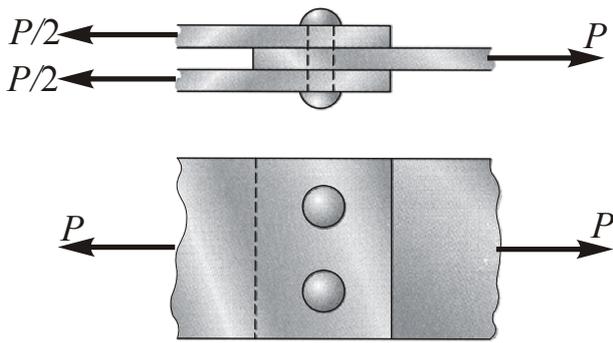
**Problem 1.45** Two lines are inscribed at right angles on a block of material. When the block is loaded in shear, the lines are found to be at an angle of  $89.75^\circ$ . What is the shear strain in the material?

**Problem 1.46** An angle bracket having thickness  $t=18$  mm is attached to the flange of a column with two 15 mm diameter bolts as shown in the figure. A uniformly distributed load acts on the top face of the bracket with a pressure  $p=2.0$  MPa. The top face of the bracket has length  $L=200$  mm and width  $b=70$  mm. Determine the bearing pressure  $\sigma_b$  between the angle

bracket and the bolts and the average shear stress  $\tau_{aver}$  in the bolts. Disregard friction between the bracket and the column.

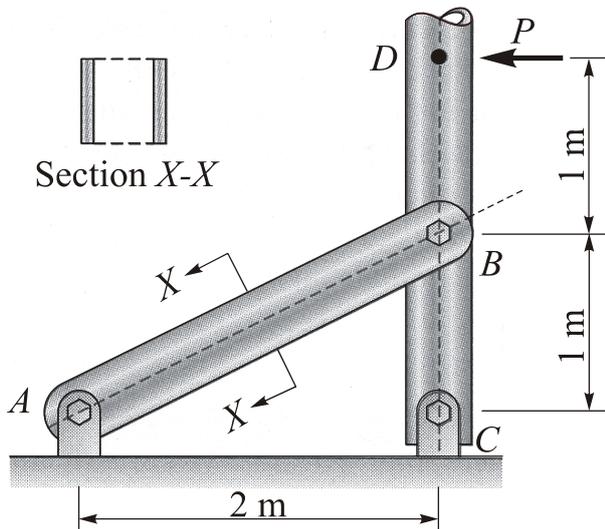
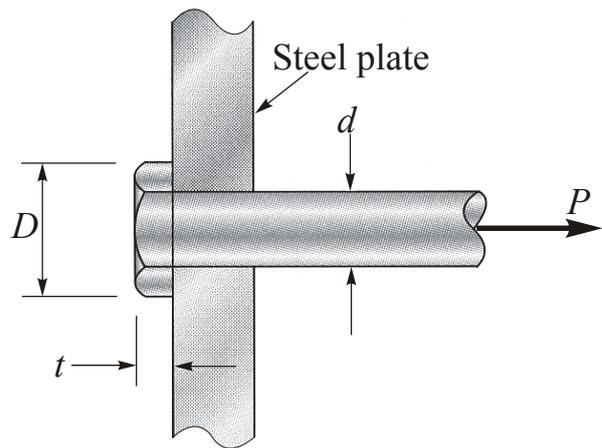


**Problem 1.47** Three steel plates, each 18 mm thick, are joined by two 16 mm rivets as shown in the figure. (1) If the load  $P=70$  kN, what is the maximum bearing stress  $\sigma_b$  on the rivets? (2) If the ultimate shear stress (average stress) in the rivets is 220 MPa, what force  $P_{ult}$  is required to cause the rivets to fail in shear? Disregard friction between the plates.



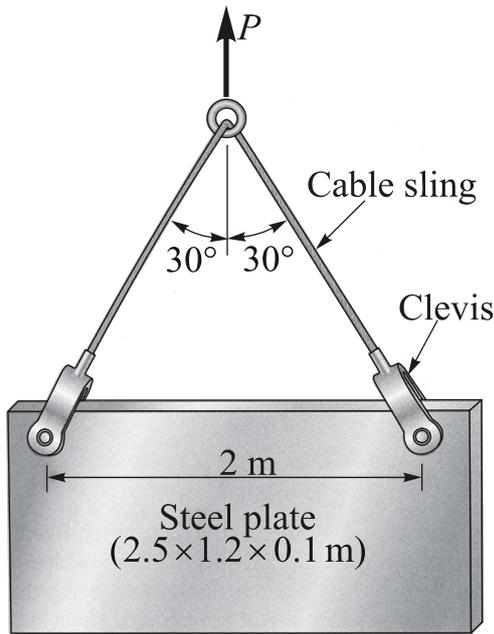
directly against the steel plate. The diameter of the circumscribed circle for the hexagon is  $D = 20$  mm (which means that each side of the hexagon has length 10 mm). Also, the thickness  $t$  of the bolt head is 6 mm. For calculation purposes, assume the tensile force  $P$  in the bolt is 5 kN. (1) Determine the average bearing stress  $\sigma_b$  between the hexagonal head of the bolt and the plate. (2) Determine the average shear stress  $\tau_{aver}$  in the head of the bolt.

**Problem 1.48** A frame  $ACD$  consists of a vertical pipe  $CD$  and a brace  $AB$  constructed from two flat bars (see figure). The frame is supported by bolted connections at points  $A$  and  $C$ , which are 2-m apart. The brace is fastened to the pipe at point  $B$  (1 m above point  $C$ ) by an 18-mm diameter bolt. A horizontal load  $P$  acts at point  $D$  (2 m above point  $C$ ). If the load  $P = 15$  kN, what is the average shear stress  $\tau_{aver}$  in the bolt at  $B$ ?

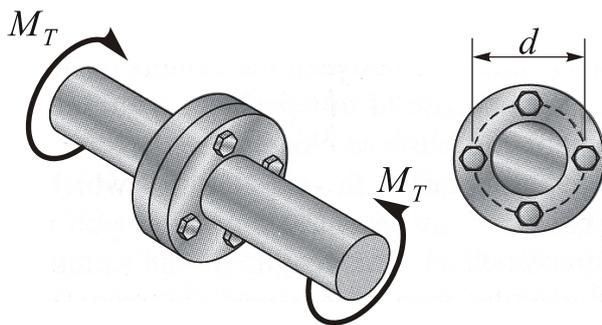


**Problem 1.50** A steel plate of dimensions  $2.5 \times 1.2 \times 0.1$  m is hoisted by a sling that has a clevis at each end (see figure). The pins through the clevises are 18 mm in diameter and located 2 m apart. Each half of the cable is at an angle of  $30^\circ$  to the vertical. For these conditions, determine the average shear stress  $\tau_{aver}$  in the pins and the bearing stress  $\sigma_b$  between the steel plate and the pins.

**Problem 1.49** A bolt of shank diameter  $d = 12$  mm passes through a hole in a steel plate (see figure). The hexagonal head of the bolt bears

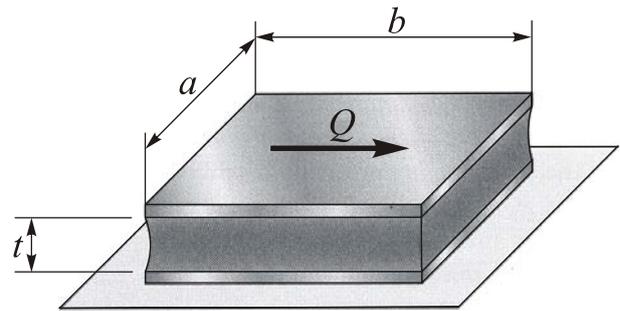


**Problem 1.51** A torque moment  $M_T$  of 8 kNm is transmitted between two flanged shafts by means of four 18-mm bolts (see figure). What is the average shear stress  $\tau_{aver}$  in each bolt if the diameter  $d$  of the bolt circle is 150 mm?

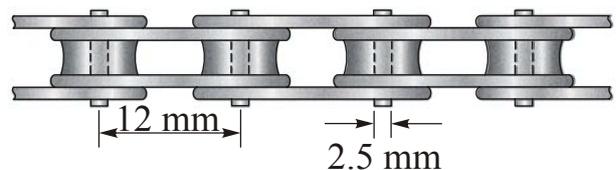


**Problem 1.52** An elastomeric bearing pad consisting of two steel plates bonded to a chloroprene elastomer is subjected to a shear force  $Q$  during a static loading test (see figure). The pad has dimensions  $a = 120$  mm and  $b = 150$  mm, and the elastomer has

thickness  $t = 40$  mm. When the force  $Q$  equals 5 kN, the top plate is found to have displaced laterally 6.0 mm with respect to the bottom plate. What is the shear modulus of elasticity  $G$  of the chloroprene?



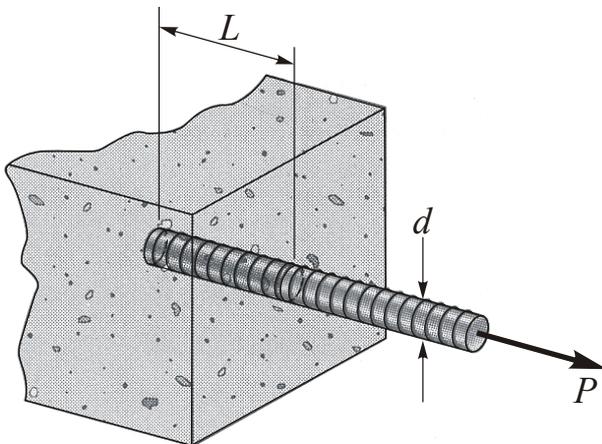
**Problem 1.53** A bicycle chain consists of a series of small links, each about 12 mm long between the centers of the pins (see figure). For the purpose of this problem, observe closely the construction of the bicycle chain links. Note particularly the cross pins, which we will assume to have a diameter of 2.5 mm. Now you must make two measurements: (1) the length of the crank arm, and (2) the radius of the chain wheel (sprocket wheel). (1) Using those dimensions, calculate the tensile force  $F$  in the chain due to a force of 800 N applied to one of the pedals. (2) Calculate the average shear stress  $\tau_{aver}$  in the pins.



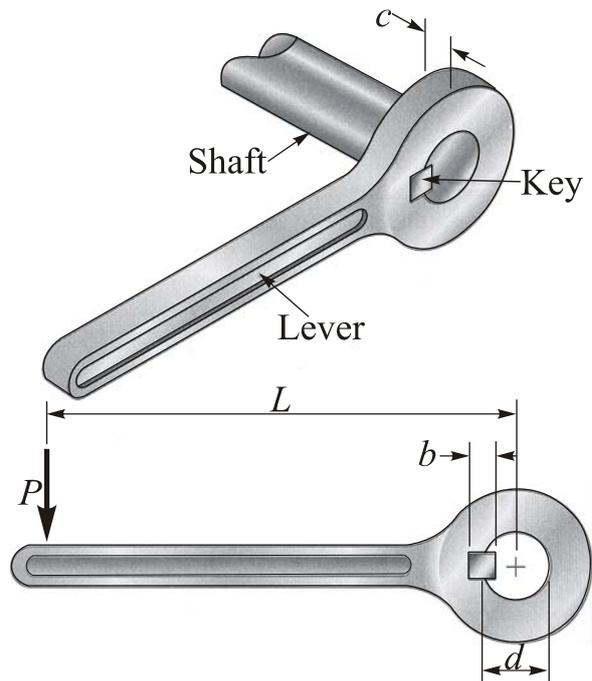
**Problem 1.54** The bond between reinforcing bars and concrete is tested by means of a “pull-out test” of a bar embedded in concrete (see figure). A tensile force  $P$  is applied to the end of the bar, which has diameter  $d$  and embedment length  $L$ . (1) Assume that the shear stress (or bond stress) between the bar and the concrete is uniformly distributed over the length  $L$ . Then, if  $P = 20 \text{ kN}$ ,  $d = 12 \text{ mm}$ , and  $L = 300 \text{ mm}$ , what average shear stress  $\tau_{aver}$  is developed between the steel and concrete? (2) In reality the bond stress between the steel and concrete is smallest near the surface and largest at the interior end of the bar. Therefore, to get slightly better (but still not very accurate) results, assume that the shear stress  $\tau$  is given by the equation

$$\tau = \frac{\tau_{max}}{4L^3} (4L^3 - 9Lx^2 + 6x^3),$$

in which  $\tau_{max}$  is the maximum shear stress and the distance  $x$  is measured from the interior end of the bar toward the surface of the concrete. Then, if  $P = 20 \text{ kN}$ ,  $d = 20 \text{ mm}$ , and  $L = 300 \text{ mm}$ , what is the maximum shear stress  $\tau_{max}$ ?

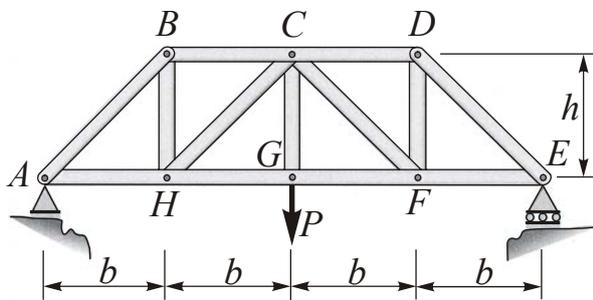


**Problem 1.55** A special lever is used to twist a circular shaft by means of a square key that fits into slots (or keyways) in the shaft and lever, as shown in the figure. The shaft has diameter  $d$ , the key has a square cross section of dimensions  $b \times b$ , and the length of the key is  $c$ . The key fits half into the lever and half into the shaft (i.e., the keyways have a depth equal to  $b/2$ ). Derive a formula for the average shear stress  $\tau_{aver}$  in the key when a load  $P$  is applied at distance  $L$  from the center of the shaft. Disregard the effects of any friction, assume that the bearing pressure between the key and the lever is uniformly distributed.

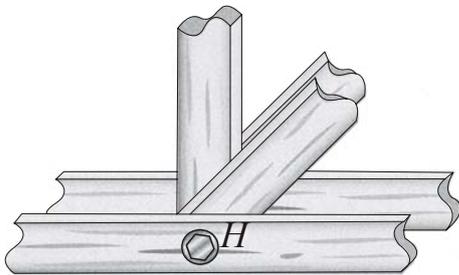


**Problem 1.56** The truss  $ABCDEFGH$  shown in the figure (a) is part of a wood bridge. The truss has height  $h$  and panel length  $b$ , with both dimensions being the same. The truss members meeting at joint  $H$  are shown

in detail in the figure (b). A single bolt of diameter  $d = 40$  mm connects the members at this joint. We will consider the effect of only one load  $P = 5$  kN acting at the midpoint (because the load has a unit value, the stresses for any other value of the load can be obtained by multiplication). (1) What is the maximum shear force  $Q_{\max}$  in the bolt at joint  $H$ ? (2) What is the average shear stress  $\tau_{aver}$  in the bolt at the cross section of maximum shear force?

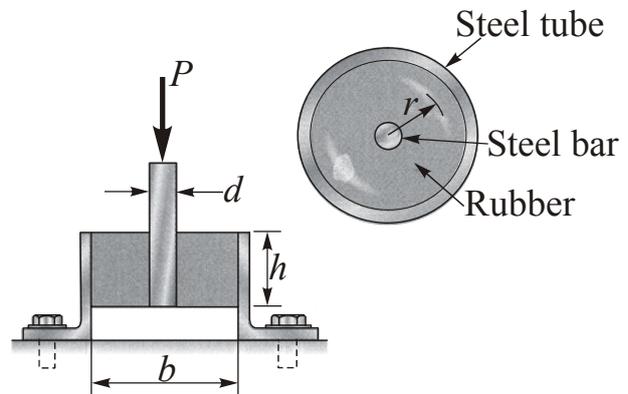


(a)



(b)

**Problem 1.57** A shock mount (shown in the figure) is used to support an expensive instrument. The mount consists of an outer steel tube with inside diameter  $b$ , a central steel bar of diameter  $d$  that supports the load  $P$ , and a hollow rubber cylinder (height  $h$ ) bonded to the tube and bar. (1) Obtain a formula for the shear stress  $\tau$  in the rubber at a radial distance  $r$  from the center of the mount. (2) Obtain a formula for the downward displacement  $\delta$  of the bar due to the load  $P$ , assuming that  $G$  is the shear modulus of elasticity of the rubber and that the steel tube and bar are rigid.



## Chapter 2 Uniaxial Stress State

### 2.1 Linear Elasticity in Tension-Compression. Hooke's Law and Poisson's Ratio. Deformability and Volume Change

Many structural materials, including most metals, wood, plastics, and ceramics, *behave both elastically and linearly* when first loaded. Consequently, their *stress-strain curves* begin with a straight line passing through the origin. An example is the stress-strain curve for *structural steel* (Fig. 2.1), where the region from the origin  $O$  to the *proportional limit* (point  $A$ ) is both linear and elastic.

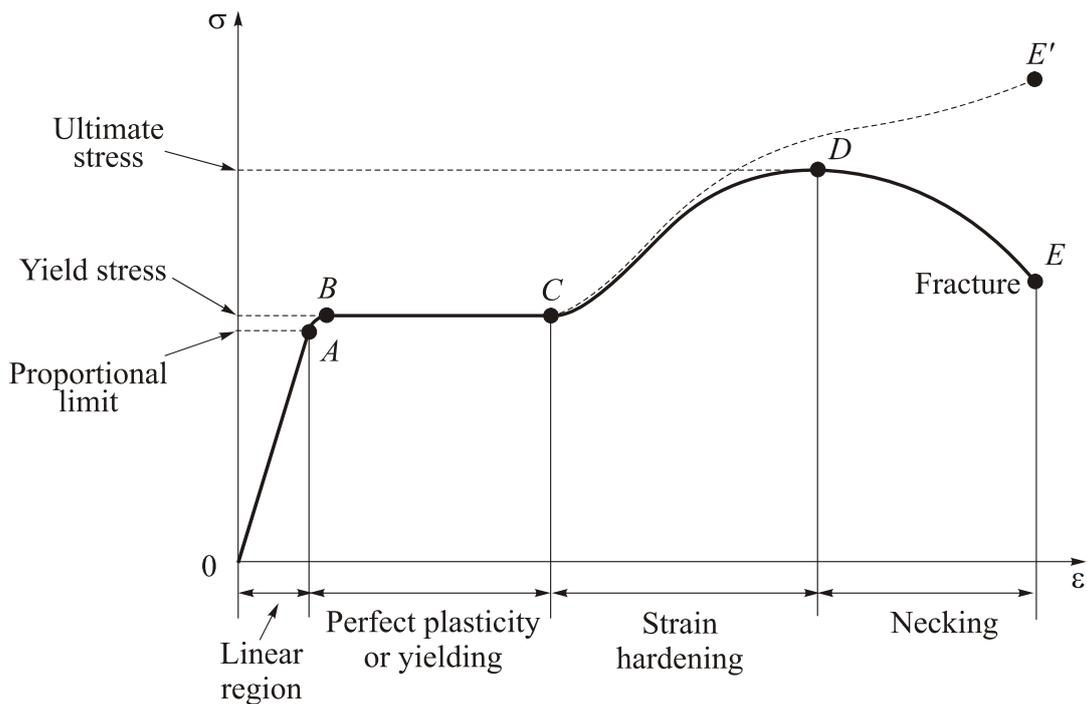


Fig. 2.1 Stress-strain diagram for a typical structural steel in tension

Other examples are the regions below both the proportional limits and the *elastic limits* on the diagrams for *aluminum* (Fig. 2.2), *rubber* (Fig. 2.3), and *copper* (Fig. 2.4).

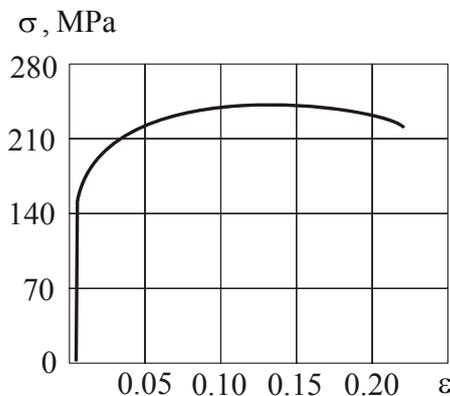


Fig. 2.2 Typical stress-strain diagram for an aluminum alloy

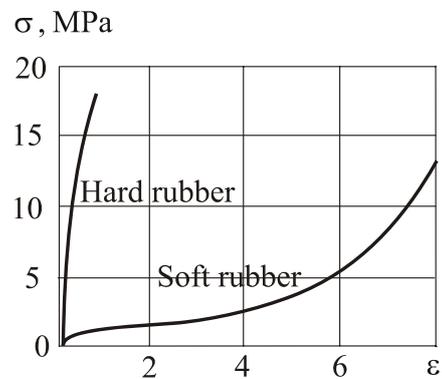


Fig. 2.3 Stress-strain curves for two kinds of rubber in tension

### 2.1.1 Hooke's law

When a material behaves elastically and also exhibits a linear relationship between stress and strain, it is said to be *linearly elastic*. This type of behavior is extremely important in engineering because in designing *structures* and *machines* to function in this region, *we will avoid permanent deformations* due to *yielding*.

The linear relationship between stress and strain for a *bar* in simple tension or compression is expressed by the equation

$$\sigma = E\varepsilon, \quad (2.1)$$

in which  $\sigma$  is the *axial stress*,  $\varepsilon$  is the *axial strain*, and  $E$  is a constant of proportionality known as the *modulus of elasticity* for the material. The modulus of elasticity is the *slope of the stress-strain diagram* in the linearly elastic region. Since strain is dimensionless, the units of  $E$  are the same as the units of stress. Typical units of  $E$  are pascals (or multiples thereof) in SI units.

The equation  $\sigma = E\varepsilon$  is commonly known as *Hooke's law*, named for the famous English scientist Robert Hooke (1635-1703). Hooke was the first person to investigate scientifically the elastic properties of materials, and he tested such diverse materials as *metal*, *wood*, *stone*, *bone*, and *sinew*. He measured the *stretching* of long *wires* supporting weights and observed that the *elongations* "always bear the same proportions one to the other that the weights do that made them". Thus, Hooke established the linear relationship between the applied loads and the resulting elongations.

Equation (2.1) is the simplest version of Hooke's law because it relates only to the longitudinal stresses and strains developed in simple tension or compression of a bar (*uniaxial stress*). To deal with more complicated states of stress, such as those found in most structures and machines, it is necessary to use more extensive equations of Hooke's law, which will be discussed below.

The modulus of elasticity as the slope of the stress-strain diagram in the linearly elastic region is different for various materials. The slope of the stress-strain diagram beyond the proportional limit is defined as the *tangent modulus*  $E_t$ , that is  $E_t = d\sigma / d\varepsilon$ . The ratio of stress to strain at any point on the curve

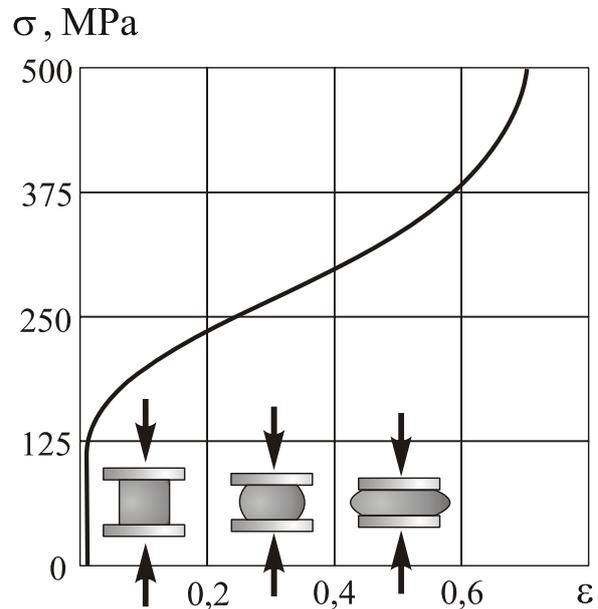


Fig. 2.4 Stress-strain diagram for copper in compression

above the proportional limit is called the *secant modulus*  $E_s$ , that is,  $E_s = \sigma / \epsilon$ . Below the proportional limit, both  $E_t$  and  $E_s$  are equal to the modulus of elasticity  $E$  (Fig. 2.5). These quantities are used as measures of the *stiffness* of material in tension or compression.

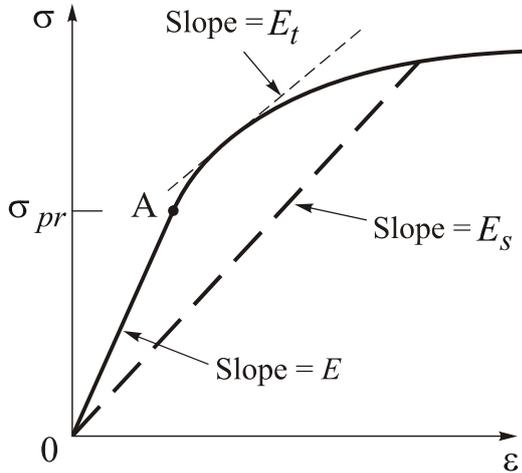


Fig. 2.5 Various moduli of elasticity

Elasticity can similarly be measured in a member subjected to shear loading. Referring to Eq. (2.1) for the linearly elastic part of the *shear stress-strain diagram*, we write

$$\tau = G\gamma. \tag{2.2}$$

Eq. (2.2) is called the *Hooke's law for shear stress and shear strain*. The constant  $G$  is termed the *modulus of rigidity*, or *shear modulus of elasticity*, of the material and is expressed in the same units as  $E$ , that is, in pascals (Pa).

The modulus of elasticity has relatively large values for materials that are very stiff, such as structural metals. Steel has a modulus of approximately 210 GPa; for aluminum, values around 73 GPa are typical. More flexible materials have a lower modulus—values for plastics range from 0.7 to 14 GPa. Some representative values of  $E$  are listed in Table 2.1.

**Table 2.1** Moduli of elasticity and Poisson's ratios

Material	Modulus of elasticity $E$	Shear modulus of elasticity $G$	Poisson's ratio, $\nu$
	GPa	GPa	
Aluminum alloys	70-79	26-30	0.33
2014-T6	73	28	0.33
6061-T6	70	26	0.33
7075-T6	72	27	0.33
Brass	96-110	36-41	0.34
Bronze	96-120	36-44	0.34
Cast iron	83-170	32-69	0.2-0.3

Material	Modulus of elasticity $E$	Shear modulus of elasticity $G$	Poisson's ratio, $\nu$
Concrete (compression)	17-31		0.1-0.2
Copper and copper alloys	110-120	40-47	0.33-0.36
Glass	48-83	19-35	0.17-0.27
Magnesium alloys	41-45	15-17	0.35
Monel (67% Ni, 30% Cu)	170	66	0.32
Nickel	210	80	0.31
Plastics			
Nylon	2.1-3.4		0.4
Polyethylene	0.7-1.4		0.4
Rock (compression) Granite, marble, Limestone	40-100 20-70		0.2-0.3 0.2-0.3
Rubber	0.0007-0.004	0.0002-0.001	0.45-0.50
Steel	190-210	75-80	0.27-0.30
Titanium alloys	100-120	39-44	0.33
Tungsten	340-380	140-160	0.2
Wood (bending) Douglas fir Oak Southern pine	11-13 11-12 11-14		

Modulus of elasticity is often called *Young's modulus*, after another English scientist, Thomas Young (1773-1829). In connection with an investigation of tension and compression of prismatic bars, Young introduced the idea of a "modulus of the elasticity." However, his modulus was not the same as the one in use today, because it involved properties of the bar as well as of the material.

### 2.1.2 Poisson's Ratio

When a prismatic bar is loaded in tension, the *axial elongation is accompanied by lateral contraction* (that is, contraction normal to the direction of the applied load). This change in shape is pictured in Fig. 2.6, where part (a) shows the bar before loading and part (b) shows it after loading. In part (b), the dashed lines represent the shape of the bar prior to loading.

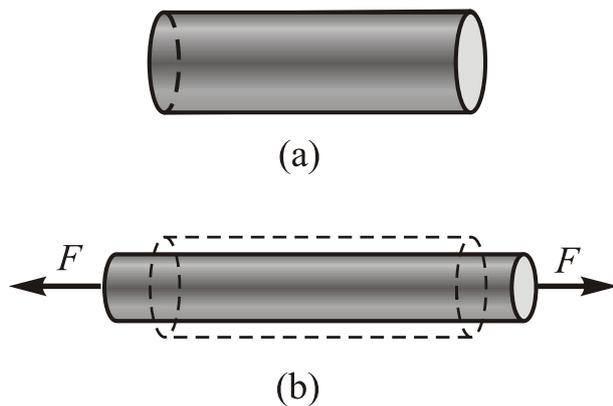


Fig. 2.6 Axial elongation and lateral contraction of a bar in tension: (a) bar before loading, and (b) bar after loading

Lateral contraction is easily seen by stretching a rubber band, but in metals the changes in lateral dimensions (in the linearly elastic region) are usually too small to be visible. However, they can be detected with sensitive measuring devices.

The *lateral strain* at any point in a bar is proportional to the axial strain at that same point if the material is linearly elastic. However, in order for the lateral strains to be the same throughout the entire bar, additional conditions must be met. First, the axial force must be constant throughout the length of the bar, so that the axial strain is constant. Second, the *material must be homogeneous*, that is, *it must have the same composition* (and hence the same elastic properties) *at every point*.

In tension experiment, we assume that the material is homogeneous so that the stress and strain would be *uniform* throughout the bar. However, it is important to recognize that having a homogeneous material does not mean that the elastic properties are the same in all directions. For instance, the modulus of elasticity could be different in the axial and lateral directions. Therefore, a third condition for uniformity in the lateral strains is that the *elastic properties must be the same in all directions perpendicular to the longitudinal axis*. Materials that are either *isotropic* or *orthotropic* meet this condition. When all three conditions are met, as is often the case, the lateral strains in a bar subjected to uniform tension will be the same at every point in the bar and the same in all lateral directions. In result, *materials having the same properties in all directions (axial, lateral, and in between) are said to be isotropic*. If the properties differ in various directions, the material is *anisotropic* (or *aeolotropic*). A special case of anisotropy occurs when the properties in a particular direction are the same throughout the material and the properties in all directions perpendicular to that direction are the same (but different from the first properties); then the material is classified as *orthotropic*. Fiber-reinforced plastics and concrete reinforced with parallel steel bars are examples of composite materials that exhibit orthotropic behavior.

The ratio of the lateral strain  $\varepsilon'$  to the axial strain  $\varepsilon$  is known as *Poisson's ratio* and is denoted by the Greek letter  $\nu$  (nu); thus,

$$\nu = -\frac{\text{lateral strain}}{\text{axial strain}} = \frac{\varepsilon'}{\varepsilon}, \quad (2.3)$$

from which

$$\varepsilon' = -\nu\varepsilon. \quad (2.4)$$

For a bar in tension, the axial strain is positive and the lateral strain is negative (because the width of the bar decreases). For compression we have the opposite situation, with the bar becoming shorter (negative axial strain) and wider (positive lateral strain). Therefore, for ordinary materials Poisson's ratio always has a positive value.

**Note. When using Eqs. 2.3 and 2.4, we must always keep in mind that they apply only to a bar in uniaxial stress, that is, a bar for which the only stress is the normal stress  $\sigma$  in the axial direction.**

Poisson's ratio is named for the famous French mathematician Simeon Denis Poisson (1781-1840), who attempted to calculate this ratio by a molecular theory of materials. For isotropic materials, Poisson found  $\nu = 0.25$ . More recent calculations based upon better models of atomic structure give  $\nu = 0.33$ . Both of these values are close to actual measured values, which are in the range 0.25 to 0.35 for most metals and many other materials. Materials with an extremely low value of Poisson's ratio include *cork*, for which  $\nu$  is practically zero, and *concrete*, for which  $\nu$  is about 0.1 or 0.2.

A table of Poisson's ratios for various materials in the linearly elastic range is given in Table 2.1. For most purposes, Poisson's ratio is assumed to be the same in both tension and compression.

When the strains in a material become large, Poisson's ratio changes. For instance, in the case of structural steel the ratio becomes almost 0.5 when plastic yielding occurs. Thus, *Poisson's ratio remains constant only in the linearly elastic range*. From a more general viewpoint, the ratio of lateral strain to axial strain is often called the *contraction ratio*. Of course, in the special case of linearly elastic behavior, the contraction ratio is the same as Poisson's ratio.

The moduli of elasticity in tension and shear are related by the following equation:

$$G = \frac{E}{2(1+\nu)}, \quad (2.5)$$

in which  $\nu$  is Poisson's ratio. This relationship shows that  $E$ ,  $G$ , and  $\nu$  are not independent elastic properties of the material. Because the value of Poisson's ratio for ordinary materials is between zero and one-half, we see from Eq. (2.5) that  $G$  must be from one-third to one-half of  $E$ .

### 2.1.3 Deformability and Volume Change

The lateral contraction of an infinitesimally small cubic element in tension is illustrated in Fig. 2.7, where it is assumed that the faces of the cube at the origin are fixed in position. The deformations shown are greatly enlarged. For the loading condition represented in the figure, we have  $\sigma_y = \sigma_z = 0$ , and  $\sigma_x$  is the axial stress. Thus the transverse strains are connected to the axial strain by Eqs. (2.1) and (2.3) as follows:

$$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\nu\frac{\sigma_x}{E}. \quad (2.6)$$

We observe from the figure that the final volume of the element is

$$V_f = (1 + \varepsilon_x)dx(1 + \varepsilon_y)dy(1 + \varepsilon_z)dz. \quad (2.7)$$

Expanding the right side and neglecting higher-order terms involving  $\varepsilon_x^2$  and  $\varepsilon_x^3$ , we obtain

$$V_f = (1 + \varepsilon_x + \varepsilon_y + \varepsilon_z)dxdydz = V_0 + \Delta V, \quad (2.8)$$

in which  $V_0$  is the *initial volume*  $dxdydz$  and  $\Delta V$  is *absolute change in volume*. The *unit volume change*  $e$  is therefore defined as

$$e = \frac{\Delta V}{V_0} = \varepsilon_x + \varepsilon_y + \varepsilon_z. \quad (2.9)$$

Substitution of Eq. (2.6) into this expression yields

$$e = (1 - 2\nu)\varepsilon_x = \frac{1 - 2\nu}{E}\sigma_x. \quad (2.10)$$

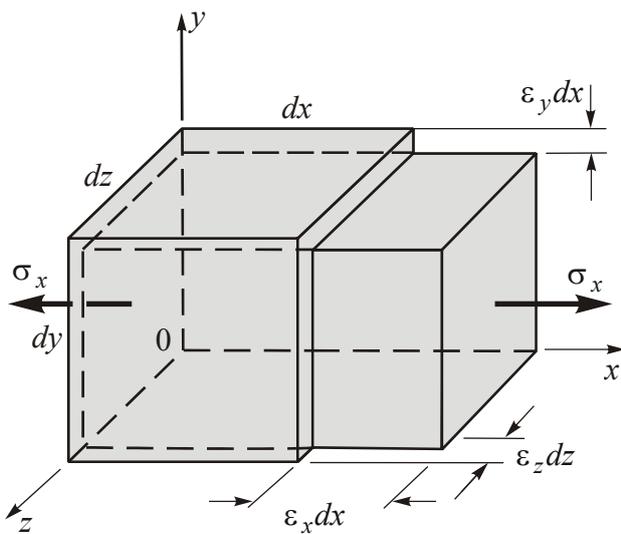


Fig. 2.7. Lateral contraction of an element of deformable solid under uniaxial tension

The quantity  $e$  is also referred to as the *dilatation*. It is observed from the foregoing result that a *tensile force increases and a compressive force decreases the volume* of the element.

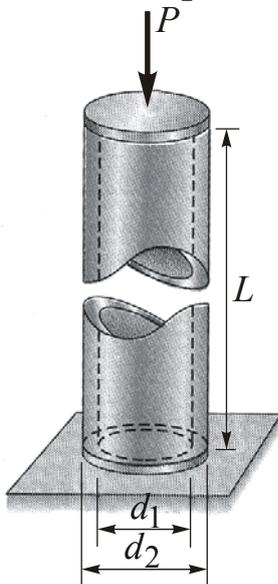
In the case of an incompressible material, we have  $e = 0$  and Eq. (2.10) shows that  $1 - 2\nu = 0$  or  $\nu = 0.5$ . For most materials in the linear elastic range  $\nu < 0.5$ , since some change in volume occurs. In the plastic region, however,

the volume remains nearly constant and hence  $\nu$  is taken as 0.5. As already pointed out, for most materials  $\nu$  is about 0.25 or 0.33 in the linearly elastic region, which means that the unit volume change is in the range  $\varepsilon/3$  to  $\varepsilon/2$ .



## EXAMPLES

### Example 2.1



A steel pipe of length  $L = 150$  cm outside diameter  $d_2 = 15$  cm, and inside diameter  $d_1 = 11$  cm is compressed by an axial force  $P = 600$  kN (see figure). The material has modulus of elasticity  $E = 210$  GPa and Poisson's ratio  $\nu = 0.30$ .

Determine the following quantities for the pipe: (1) the shortening  $\delta$ ; (2) the lateral strain  $\varepsilon'$ ; (3) the increase  $\Delta d_2$  in the outer diameter and the increase  $\Delta d_1$  in the inner diameter; (4) the increase  $\Delta t$  in the wall thickness; (5) the increase  $\Delta V$  in the volume of material, and (6) the dilatation  $e$ .

**Solution** The cross-sectional area  $A$  and longitudinal stress  $\sigma$  are determined as follows:

$$A = \frac{\pi}{4} (d_2^2 - d_1^2) = \frac{\pi}{4} [(15 \text{ cm})^2 - (11 \text{ cm})^2] = 81.64 \times 10^{-4} \text{ cm}^2,$$

$$\sigma = \frac{P}{A} = \frac{-600 \times 10^3 \text{ N}}{81.64 \times 10^{-4} \text{ cm}^2} = -73.5 \text{ MPa (compression)}.$$

Because the stress is well below the yield stress for steel (from 200 MPa for structural steels up to 1,600 MPa for spring steels), the material behaves linearly elastically and the axial strain may be found from Hooke's law:

$$\varepsilon = \frac{\sigma}{E} = \frac{-73.5 \times 10^6}{210 \times 10^9} = -0.35 \times 10^{-3}.$$

(1) Knowing the axial strain, we can now find the change in length of the pipe:

$$\delta = \varepsilon L = (-0.35 \times 10^{-3})(1.2 \text{ m}) = -0.42 \times 10^{-3} \text{ m} = -0.42 \text{ mm}.$$

The negative sign for  $\delta$  indicates a shortening of the pipe.

(2) The lateral strain is obtained from Poisson's ratio  $\varepsilon' = -\nu\varepsilon$  :

$$\varepsilon' = -\nu\varepsilon = -(0.30)(-0.35 \times 10^{-3}) = 0.105 \times 10^{-3}.$$

The positive sign for  $\varepsilon'$  indicates an increase in the lateral dimensions, as expected for compression.

(3) The increase in outer diameter equals the lateral strain times the diameter:

$$\Delta d_2 = \varepsilon' d_2 = (0.105 \times 10^{-3})(15 \text{ cm}) = 1.575 \times 10^{-5} \text{ m}.$$

Similarly, the increase in inner diameter is

$$\Delta d_1 = \varepsilon' d_1 = (0.105 \times 10^{-3})(11 \text{ cm}) = 1.155 \times 10^{-5} \text{ m}.$$

(4) The increase in wall thickness is found in the same manner as the increases in the diameters; thus,

$$\Delta t = \varepsilon' t = (0.105 \times 10^{-3})(2 \text{ cm}) = 0.21 \times 10^{-5} \text{ m}.$$

This result can be verified by noting that the increase in wall thickness is equal to the following:

$$\Delta t = \frac{\Delta d_2 - \Delta d_1}{2} = \frac{1}{2}(1.575 - 1.155) \times 10^{-5} \text{ m} = 0.21 \times 10^{-5} \text{ m}, \text{ as expected.}$$

**Note. Under compression, all three quantities increase (outer diameter, inner diameter, and thickness).**

(5) The change in volume of the material is calculated from:

$$\begin{aligned} \Delta V &= V_f - V_0 = V_0 \varepsilon (1 - 2\nu) = AL\varepsilon (1 - 2\nu) = \\ &= (81.64 \times 10^{-8} \text{ m}^2)(1.2 \text{ m})(-0.35 \times 10^{-3})(1 - 0.60) = -14.683 \times 10^{-7} \text{ m}^3. \end{aligned}$$

The volume change is negative, indicating a decrease in volume, as expected for compression.

(6) Finally, the dilatation is

$$\begin{aligned} e &= \frac{\Delta V}{V_0} = \varepsilon (1 - 2\nu) = \frac{\sigma}{E} (1 - 2\nu), \\ e &= \varepsilon (1 - 2\nu) = (-0.35 \times 10^{-3})(1 - 0.60) = -0.00014, \end{aligned}$$

which is a 0.014% reduction in the volume of material.

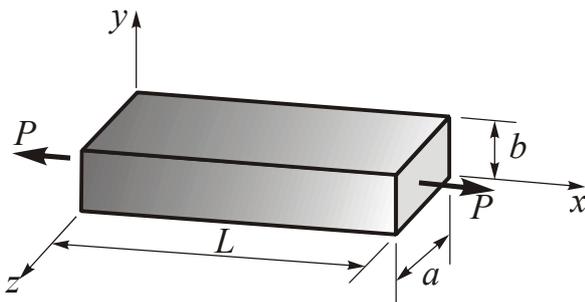
**Note. The numerical results obtained in this example illustrate that the dimensional changes in structural materials under normal loading conditions**

are extremely small. In spite of their smallness, changes in dimensions can be important in certain kinds of analysis (such as the analysis of statically indeterminate structures) and in the experimental determination of stresses and strains.

### Example 2.2

A steel rectangular block,  $a = 2.4$  cm wide and  $b = 1.2$  cm deep, is subjected to an axial tensile load as shown in the figure. Measurements show the block to increase in length by  $\delta_x = 7.11 \times 10^{-5}$  m (initial length  $L = 10$  cm) and to decrease in width by  $\delta_z = 0.533 \times 10^{-5}$  m, when  $P$  is 45 kN. Calculate the modulus of elasticity and Poisson's ratio for the material.

**Solution** The cross-sectional area of the block is  $A = 2.4 \times 1.2 = 2.88 \text{ cm}^2$ . The axial stress and strain are



$$\sigma_x = \frac{P}{A} = \frac{45 \times 10^3}{2.88 \times 10^{-4}} = 156.3 \text{ MPa},$$

$$\varepsilon_x = \frac{\delta_x}{L} = \frac{7.11 \times 10^{-5}}{10^{-1}} = 7.11 \times 10^{-4}.$$

The transverse strain in the  $z$  direction is

$$\varepsilon_z = \frac{\delta_z}{a} = -\frac{0.533 \times 10^{-5}}{2.4 \times 10^{-2}} = -2.22 \times 10^{-4}.$$

The depth of the block contracts by  $\delta_y = (2.22 \times 10^{-4})(1.2 \times 10^{-2}) = 2.664 \times 10^{-6}$  m, since  $\varepsilon_y = \varepsilon_z$ . Formulas (2.1) and (2.3) result in the values

$$E = \frac{\sigma_x}{\varepsilon_x} = \frac{156.3 \times 10^6}{7.11 \times 10^{-4}} = 219.8 \times 10^9 \text{ Pa} = 219.8 \text{ GPa}$$

and

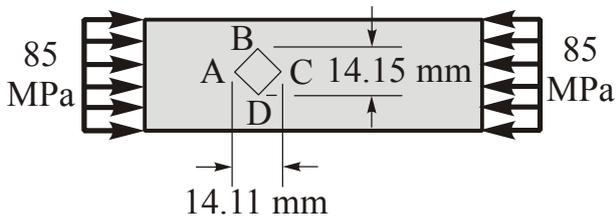
$$\nu = -\frac{\varepsilon_y}{\varepsilon_x} = \frac{2.22 \times 10^{-4}}{7.11 \times 10^{-4}} = 0.31$$

for the modulus of elasticity and Poisson's ratio, respectively.



**PROBLEMS**

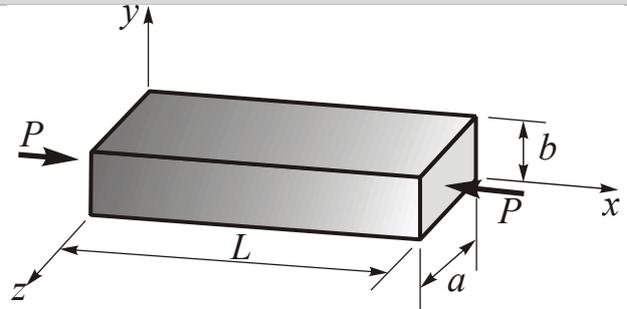
**Problem 2.1** A 10-mm by 10-mm square  $ABCD$  is drawn on a member prior to loading. After loading, the square becomes the rhombus shown in the figure. Determine (1) the modulus of elasticity and (2) Poisson's ratio.



**Problem 2.2** A 50.8 mm-diameter bar 1.83 m long shortens 1.19 mm under an axial load of 178 kN. If the diameter is increased 0.01 mm during loading, calculate (1) Poisson's ratio; (2) the modulus of elasticity; and (3) the shear modulus of elasticity.

**Problem 2.3** A round steel rod of diameter 25 mm is subjected to axial tensile force  $F$ . The decrease in diameter is 0.012 mm. Compute the largest value of  $F$  for  $E = 200$  GPa and  $\nu = 0.33$ .

**Problem 2.4** The block is subjected to an axial compression of  $P = 400$  kN (see figure). Use  $a = 60$  mm,  $b = 40$  mm, and  $L = 200$  mm. If dimensions  $b$  and  $L$  are changed to 40.02 and 199.7 mm, respectively, calculate: (1) Poisson's ratio; (2) the modulus of elasticity; (3) the final value of the dimension  $a$ ; and (4) the shear modulus of elasticity.



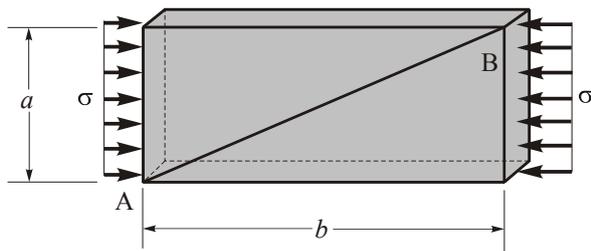
**Problem 2.5** Calculate the smallest diameter and shortest length that may be selected for a steel control rod of a machine under an axial load of 4 kN if the rod must stretch 2.5 mm. Use  $E = 200$  GPa and  $\sigma_{all} = 150$  MPa.

**Problem 2.6** A prismatic bar is loaded in tension by axial forces. Find Poisson's ratio for the material if the ratio of the unit volume change to the unit change in cross-sectional area is equal to  $-0.67$ .

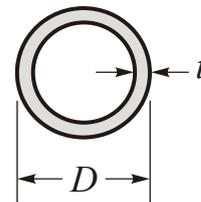
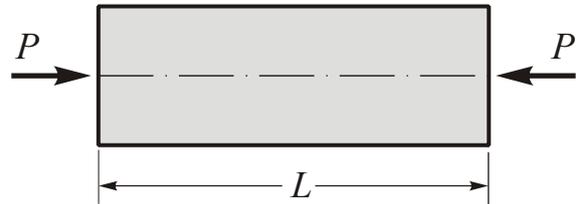
**Problem 2.7** Verify that the change in the slope of the diagonal line  $AB$ ,  $\Delta$ , of a rectangular plate (see figure) subjected to a uniaxial compression stress  $\sigma$  is given by

$$\Delta = \frac{a}{b} \left[ \frac{1 + (\nu\sigma/E)}{1 - (\sigma/E)} - 1 \right],$$

where  $a/b$  is the initial slope. For  $a = 25$  mm,  $b = 50$  mm,  $\nu = 0.3$ , and  $E = 70$  GPa, calculate the value of  $\Delta$  when  $\sigma = 120$  MPa.



compressive load  $P = 200$  kN. Determine the change in (1) length  $\Delta L$ ; (2) diameter  $\Delta D$ ; and (3) thickness  $\Delta t$ .



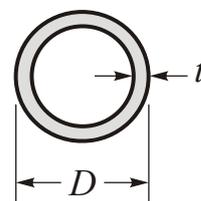
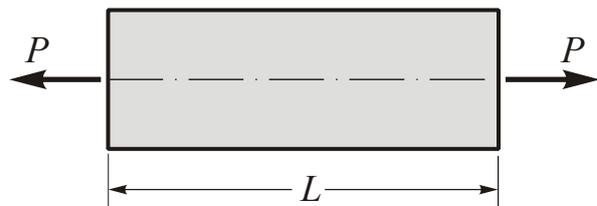
**Problem 2.8** A 15-mm-diameter bar with a 125-mm-gage length is subjected to a gradually increasing tensile load. At the proportional limit, the value of the load is 35 kN, the gage length increases 0.35 mm, and the diameter decreases 0.015 mm. Calculate (1) the proportional limit; (2) the modulus of elasticity; (3) Poisson's ratio; and (4) the shear modulus of elasticity.

**Problem 2.9** A 6-m-long truss member is made of two 50-mm-diameter steel bars ( $E = 210$  GPa,  $\sigma_{pr} = 230$  MPa,  $\nu = 0.3$ ). Given a tensile load of 300 kN, calculate the change in (1) the length of the member and (2) the diameter of a bar.

**Problem 2.10** A 5-cm-diameter solid brass bar ( $E = 103$  GPa,  $\nu = 0.3$ ) is fitted in a hollow bronze tube. Determine the internal diameter of the tube so that its surface and that of the bar are just in contact, with no pressure, when the bar is subjected to an axial compressive load  $P = 180$  kN.

**Problem 2.11** The cast-iron pipe shown in the figure ( $E = 70$  GPa,  $\nu = 0.3$ ), which has length  $L = 0.5$  m, outside diameter  $D = 150$  mm, and wall thicknesses  $t = 15$  mm, is under an axial

**Problem 2.12** The brass pipe shown in the figure ( $E = 105$  GPa,  $\nu = 0.3$ ), which has length  $L = 0.5$  m, outside diameter  $D = 150$  mm, and wall thicknesses  $t = 15$  mm, is under an axial tensile load  $P = 200$  kN. Determine the change in (1) length  $\Delta L$ ; (2) diameter  $\Delta D$ ; and (3) thickness  $\Delta t$ .



**Problem 2.13** The aluminum rod, 50 mm in diameter and 1.2 m in length, of a hydraulic ram is subjected to the maximum axial loads

of  $\pm 200$  kN. What are the largest diameter and the largest volume of the rod during service? Use  $E = 70$  GPa and  $\nu = 0.3$ .

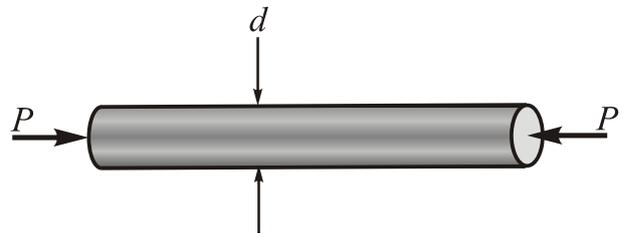
**Problem 2.14** A 20-mm-diameter bar is subjected to tensile loading. The increase in length resulting from the load of 50 kN is 0.2 mm for an initial length of 100 mm. Determine (1) the conventional and true strains and (2) the modulus of elasticity.

**Problem 2.15** A 25-mm-diameter solid aluminum-alloy bar ( $E = 70$  GPa and  $\nu = 0.3$ ) is fitted in a hollow plastic tube of 25.05 mm internal diameter. Determine the maximum axial compressive load that can be applied to the bar for which its surface and that of the tube are just in contact and under no pressure.

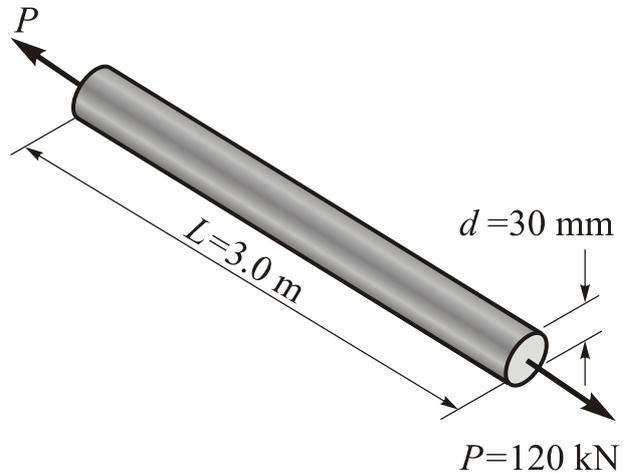
**Problem 2.16** A cast-iron bar ( $E = 80$  GPa,  $\nu = 0.3$ ) of diameter  $d = 75$  mm and length  $L = 0.5$  m is subjected to an axial compressive load  $P = 200$  kN. Determine (1) the increase  $\Delta d$  in diameter; (2) the decrease  $\Delta L$  in length; and (3) the change in volume  $\Delta V$ .

**Problem 2.17** A 50-mm-diameter and 100-mm-long solid cylinder is subjected to uniform axial tensile stresses of  $\sigma_x = 50$  MPa. Use  $E = 205$  MPa and  $\nu = 0.33$ . Calculate (1) the change in length of the cylinder and (2) the change in volume of the cylinder.

**Problem 2.18** A high-strength steel rod having modulus of elasticity  $E = 200$  GPa and Poisson's ratio  $\nu = 0.29$  is compressed by an axial force  $P$  (see figure). Before the load was applied, the diameter of the rod was exactly 25.000 mm. In order to provide certain clearances, the diameter of the rod must not exceed 25.025 mm under load. What is the largest permissible load  $P$ ?



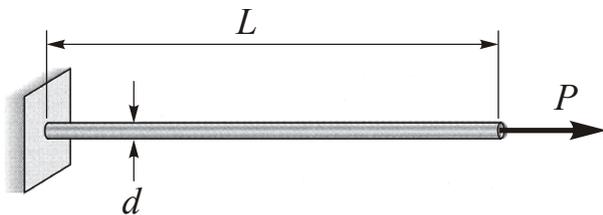
**Problem 2.19** A prismatic bar of circular cross section is loaded by tensile forces  $P = 120$  kN (see figure). The bar has length  $L = 3.0$  m and diameter  $d = 30$  mm. It is made of aluminum alloy with modulus of elasticity  $E = 73$  GPa and Poisson's ratio  $\nu = 0.33$ . Calculate (1) the elongation  $\delta$ ; (2) the decrease in diameter  $\Delta d$ , and (3) the increase in volume  $\Delta V$  of the bar.



**Problem 2.20** A high-strength steel wire, 3 mm in diameter, stretches 35 mm when a 15 m length of it is stretched by a force of 4 kN. (1) What is the modulus of elasticity  $E$  of the steel? (2) If the diameter of the wire decreases by  $2.2 \times 10^{-3}$  mm, what is Poisson's ratio? (3) What is the unit volume change for the steel?

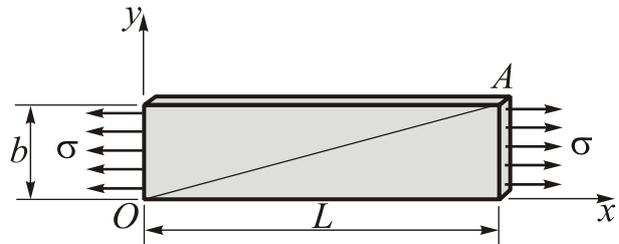
**Problem 2.21** A round bar of 10 mm diameter is made of aluminum alloy. When the bar is stretched by an axial force  $P$ , its diameter decreases by 0.016 mm. Find (1) the magnitude of the load  $P$  and (2) the dilatation of the bar.

**Problem 2.22** A bar of monel metal (length  $L = 200$  mm, diameter  $d = 6$  mm) is loaded axially by a tensile force of  $P = 6.7$  kN (see figure). Using the data ( $E = 170$  GPa,  $\nu = 0.32$ ), determine (1) increase in length; (2) decrease in diameter of the bar; (3) increase in volume of the bar and (4) the dilatation.

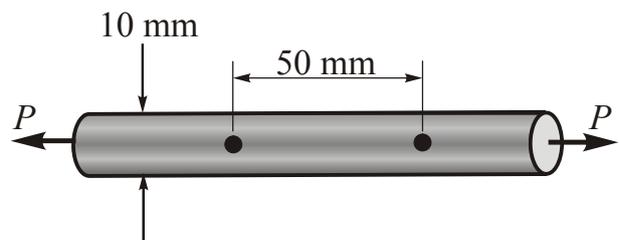


**Problem 2.23** A plate of length  $L$  and width  $b$  is subjected to a uniform tensile stress  $\sigma$  at the ends (see figure). The material has modulus of elasticity  $E$  and Poisson's ratio  $\nu$ . Before the stress was applied, the slope of the diagonal

line  $OA$  was  $b/L$ . (1) What is the slope when the stress  $\sigma$  is acting? (2) What is the unit change in area of the face of the plate? (3) What is the unit change in cross-sectional area?



**Problem 2.24** A tensile test is performed on a brass specimen 10 mm in diameter using a gage length of 50 mm (see figure). When the tensile load  $P$  reaches a value of 20 kN, the distance between the gage marks has increased by 0.122 mm. (1) What is the modulus of elasticity  $E$  of the brass? (2) If the diameter decreases by 0.00830 mm, what is Poisson's ratio? (3) What is the dilatation of the bar?



**Problem 2.25** Derive a formula for the increase  $\Delta V$  in the volume of a prismatic bar of length  $L$  hanging vertically under its own weight ( $W$  – total weight of the bar).

## 2.2 Stresses on Inclined Planes in Uniaxial Stress State

In our previous discussions of tension and compression in axially loaded members, the only stresses we considered were the normal stresses acting on cross sections. These stresses are pictured in Fig. 2.8, where we consider a bar  $AB$  subjected to axial load  $P$ . When the bar is cut at an intermediate cross section by a plane  $m-n$  (perpendicular to the  $x$  axis), we obtain the *free-body diagram* shown in Fig. 2.8b. The normal stresses acting over the cut section may be calculated from the formula  $\sigma_x = P/A$  provided that the stress distribution is uniform over the entire cross-sectional area  $A$ . This assumption may be used if the *bar is prismatic, the material is homogeneous, the axial force  $P$  acts at the centroid of the cross-sectional area, and the cross section is away from any localized stress concentrations (Saint-Venant's zones)*. Because the cross-section is perpendicular to the longitudinal axis of the bar, there are no shear stresses acting on the cut section. In Fig. 2.8c the stresses in a two-dimensional view of the bar are shown.

The most useful way of representing the stresses in the bar of Fig. 2.8 is to

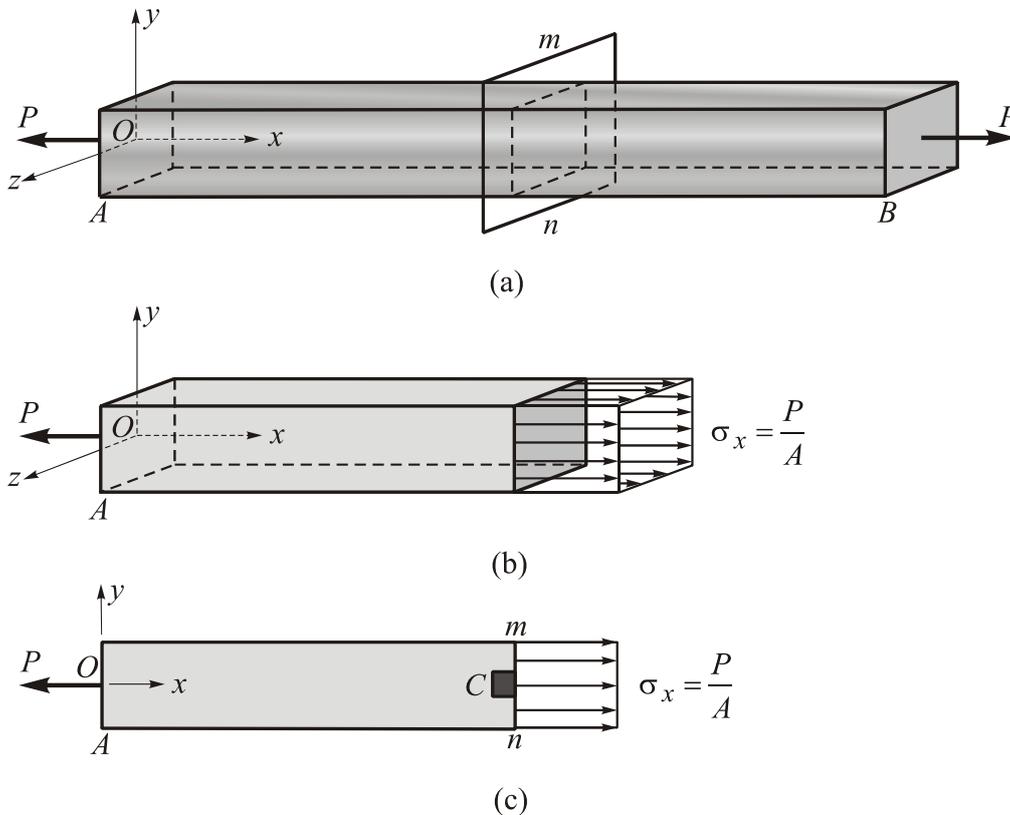


Fig. 2.8 Prismatic bar tension showing the stresses acting on cross section  $m-n$ : (a) bar with axial forces  $P$ , (b) three-dimensional view of the cut bar showing the normal stresses, and (c) two-dimensional view

isolate a small element of material, such as the element labeled C in Fig. 2.8, and then show the stresses acting on all faces of this element. An element of this kind is called a *stress element*. The stress element at point C is a small rectangular block (it doesn't matter whether it is a cube or a rectangular parallelepiped) with its right-hand face lying in cross section  $m-n$ . The dimensions of any stress element are assumed to be infinitesimally small (Fig. 2.9a). In this case, the edges of the element are parallel to the  $x$ ,  $y$ , and  $z$  axes, and the only stresses are the normal stresses  $\sigma_x$  acting on the  $x$  faces (recall that the  $x$  faces have their normals parallel to the  $x$  axis). Two-dimensional view of the element is represented in Fig. 2.9b.

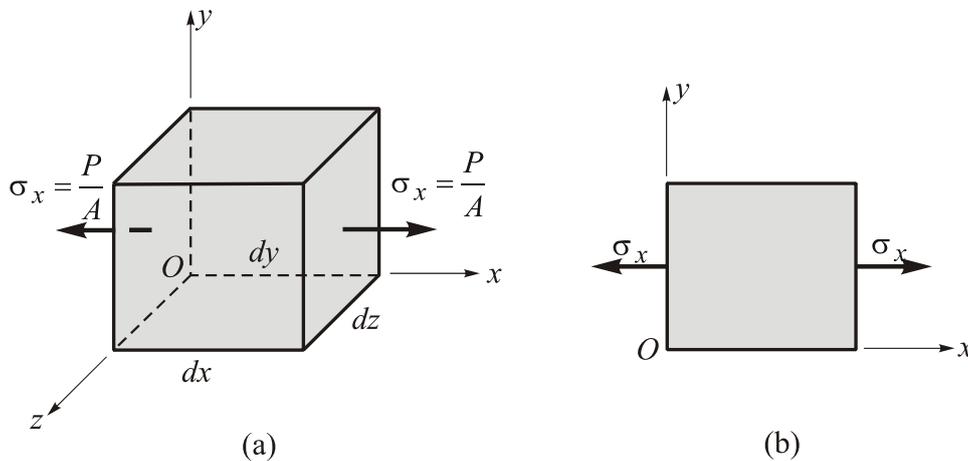


Fig. 2.9 Stress element at point of the axially loaded bar shown in Fig 2.8c: (a) three-dimensional view of the element, and (b) two-dimensional view of the element

The stress element of Fig. 2.9 provides only a limited view of the stresses in an axially loaded bar. To obtain a more complete picture, we need to investigate the stresses acting on *inclined planes*, such as the section cut by the inclined plane  $p-q$  in Fig. 2.10a. The stresses acting over the inclined section must be uniformly distributed. It is shown in the free-body diagrams of Fig. 2.10b (three-dimensional view) and Fig. 2.10c (two-dimensional view). From the equilibrium of the free body we know that the resultant of the stresses must be a horizontal force  $P$  (dashed line in Figs. 2.10b and 2.10c).

For specifying the orientation of the inclined section  $p-q$  we install the angle  $\theta$  between the  $x$  axis and the normal  $n$  to the section (Fig. 2.11a). By contrast, cross section  $m-n$  (Fig. 2.8a) has an angle  $\theta$  equal to zero (because the normal to the section is the  $x$  axis). For the stress element of Fig. 2.9 the angle  $\theta$  for the right-hand face is  $0$ , for the top face is  $90^\circ$  (a longitudinal section of the bar), for the left-hand face is  $180^\circ$ , and for the bottom face is  $270^\circ$  (or  $-90^\circ$ ).

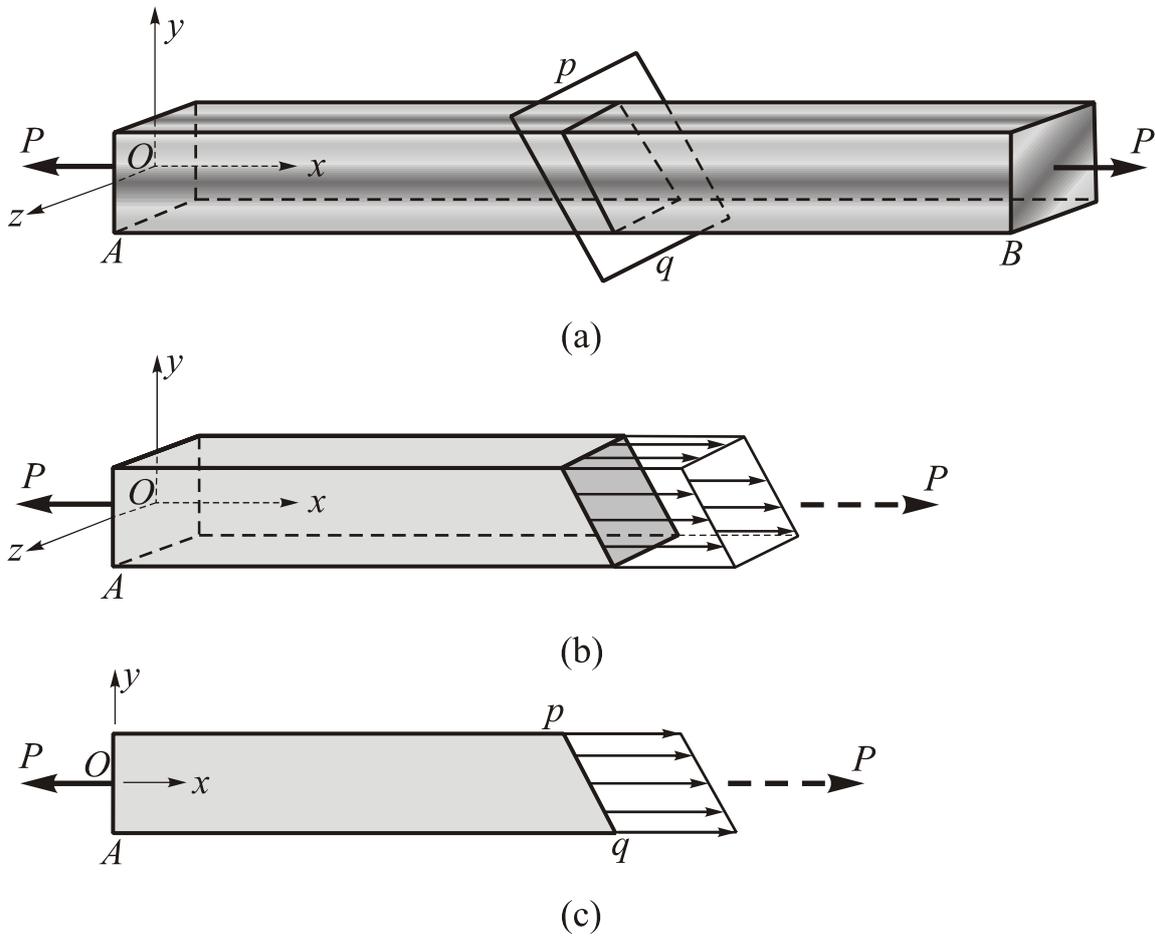


Fig. 2.10 Prismatic bar in tension showing the stresses acting on an inclined section  $p-q$ : (a) bar with axial forces  $P$ , (b) three-dimensional view of the cut bar showing the stresses, and (c) two-dimensional view

To find the stresses acting on section  $p-q$  (Fig. 2.11b) we will use the fact that the resultant of these stresses is a force  $P$  acting in the  $x$  direction. This resultant may be resolved into two components, a normal force  $N$  that is perpendicular to the inclined plane  $p-q$  and a shear force  $Q$  that is tangential to it. These force components are

$$N = P \cos \theta, \quad Q = P \sin \theta. \quad (2.11 \text{ a, b})$$

In assumption of uniform stress distribution the normal stress is equal to the normal force  $N$  divided by the area of the section, and the shear stress is equal to the shear force  $Q$  divided by the area of the section. Thus, the stresses are

$$\sigma = \frac{N}{A_1}, \quad \tau = \frac{Q}{A_1}, \quad (2.12 \text{ a, b})$$

in which  $A_1$  is the area of the inclined section ( $A$  represents the cross-sectional area of the bar):

$$A_1 = \frac{A}{\cos \theta}. \quad (2.13)$$

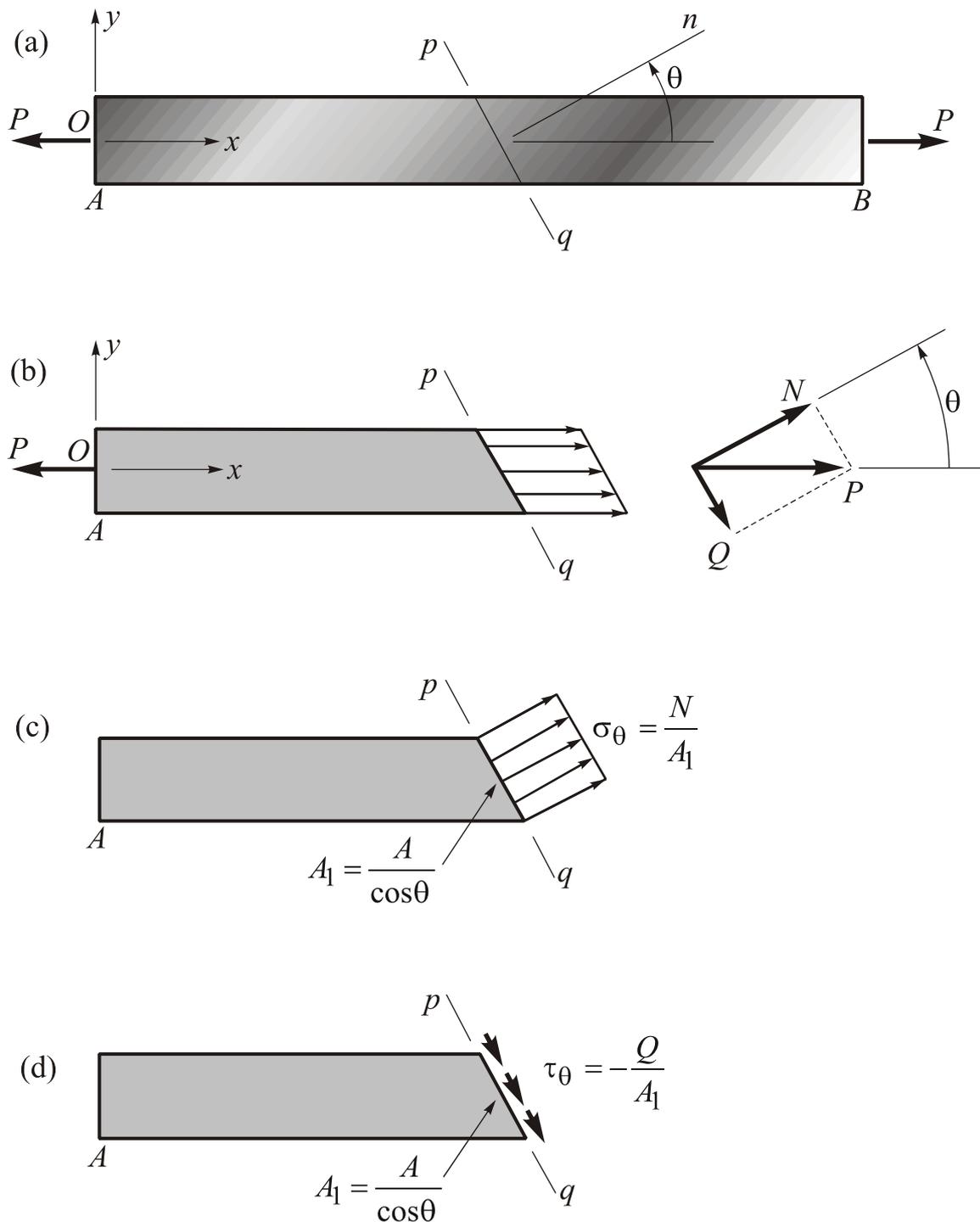


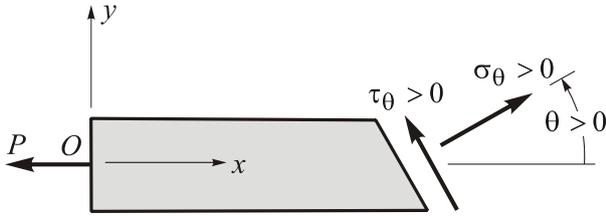
Fig. 2.11 Prismatic bar in tension showing the stresses acting on an inclined section  $p-q$

The stresses  $\sigma$  and  $\tau$  act in the directions shown in Figs. 2.11c and d, that is, in the same directions as the normal force  $N$  and shear force  $Q$ , respectively.

We need to establish now a standardized *sign convention* for stresses acting on inclined sections. We will use a subscript  $\theta$  to indicate that the stresses act on a section inclined at an angle  $\theta$  (Fig. 2.12), just as we use a subscript  $x$  to indicate

that the stresses act on a section perpendicular to the  $x$  axis (see Fig. 2.8). *Normal stresses  $\sigma_\theta$  are positive in tension and shear stresses  $\tau_\theta$  are positive when they tend to produce counterclockwise rotation of the material, as shown in Fig. 2.12.*

For a bar in tension, the normal force  $N$  produces positive normal stresses  $\sigma_\theta$  (see Fig. 2.11c) and the shear force  $Q$  produces negative shear stresses  $\tau_\theta$  (see Fig. 2.11d). These stresses are given by the following equations:



$$\sigma_\theta = \frac{N}{A_1} = \frac{P}{A} \cos^2 \theta, \quad (2.14)$$

$$\tau_\theta = -\frac{Q}{A_1} = -\frac{P}{A} \cos \theta \sin \theta. \quad (2.15)$$

Fig. 2.12 Sign convention for stresses acting on an inclined section (normal stresses are positive when are tensiled and shear stresses are positive when they tend to produce counterclockwise rotation)

Introducing the notation  $\sigma_x = P/A$ , in which  $\sigma_x$  is the normal stress on a cross section, and also using the trigonometric relations

$$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta), \quad \sin \theta \cos \theta = \frac{1}{2} \sin 2\theta, \quad (2.16)$$

we get

$$\sigma_\theta = \sigma_x \cos^2 \theta = \frac{\sigma_x}{2}(1 + \cos 2\theta), \quad (2.17a)$$

$$\tau_\theta = -\sigma_x \sin \theta \cos \theta = -\frac{\sigma_x}{2}(\sin 2\theta). \quad (2.17b)$$

These equations give the normal and shear stresses acting on an inclined section oriented at an angle  $\theta$ .

**Note.** Eqs. (2.17a) and (2.17b) were derived only from statics, and therefore they are independent of the material. Thus, these equations are valid for any material, whether it behaves linearly or nonlinearly, elastically or inelastically.

The graphs, which show the stresses variation as the inclined section is cut at various angles are shown in Fig. 2.13. The horizontal axis gives the angle as it varies from  $-90^\circ$  to  $+90^\circ$ , and the vertical axis gives the stresses  $\sigma_\theta$  and  $\tau_\theta$ .

**Note.** A positive angle is measured counterclockwise from the  $x$  axis (Fig. 2.12) and a negative angle is measured clockwise.

As shown on the graph, the normal stress  $\sigma_\theta$  equals  $\sigma_x$  when  $\theta = 0$ . Then, as  $\theta$  increases or decreases, the normal stress diminishes until at  $\theta = \pm 90^\circ$  it becomes zero, because there are no normal stresses on sections cut parallel to the longitudinal axis. The *maximum normal stress* occurs at  $\theta = 0$  and is

$$\sigma_{\max} = \sigma_x. \quad (2.18)$$

When  $\theta = \pm 45^\circ$ , the normal stress is one-half the maximum value.

The shear stress  $\tau_\theta$  is zero on cross sections of the bar ( $\theta = 0$ ) as well as on longitudinal sections ( $\theta = \pm 90^\circ$ ). Between these extremes, the stress varies as shown on the graph, reaching the largest positive value when  $\theta = -45^\circ$  and the largest negative value when  $\theta = +45^\circ$ . These *maximum shear stresses* have the same magnitude:

$$\tau_{\max} = \frac{\sigma_x}{2}, \quad (2.19)$$

but they tend to rotate the element in opposite directions.

The maximum stresses in a bar in tension are shown in Fig. 2.14. Two stress elements are selected. Element *A* is oriented at  $\theta = 0^\circ$  and element *B* is oriented at  $\theta = 45^\circ$ . Element *A* has the maximum normal stresses (Eq. 2.18) and element *B* has the maximum shear stresses (Eq. 2.19). In the case of element *A* (Fig. 2.14b), *the only stresses are the maximum normal stresses without shear stresses on any of the faces.*

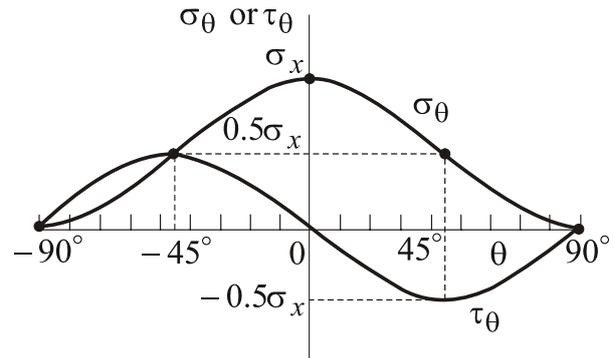


Fig. 2.13 Graph of normal stress  $\sigma_\theta$  and shear stress  $\tau_\theta$  versus angle of the inclined section (see Fig. 2.12 and Eqs. 2.17a and b)

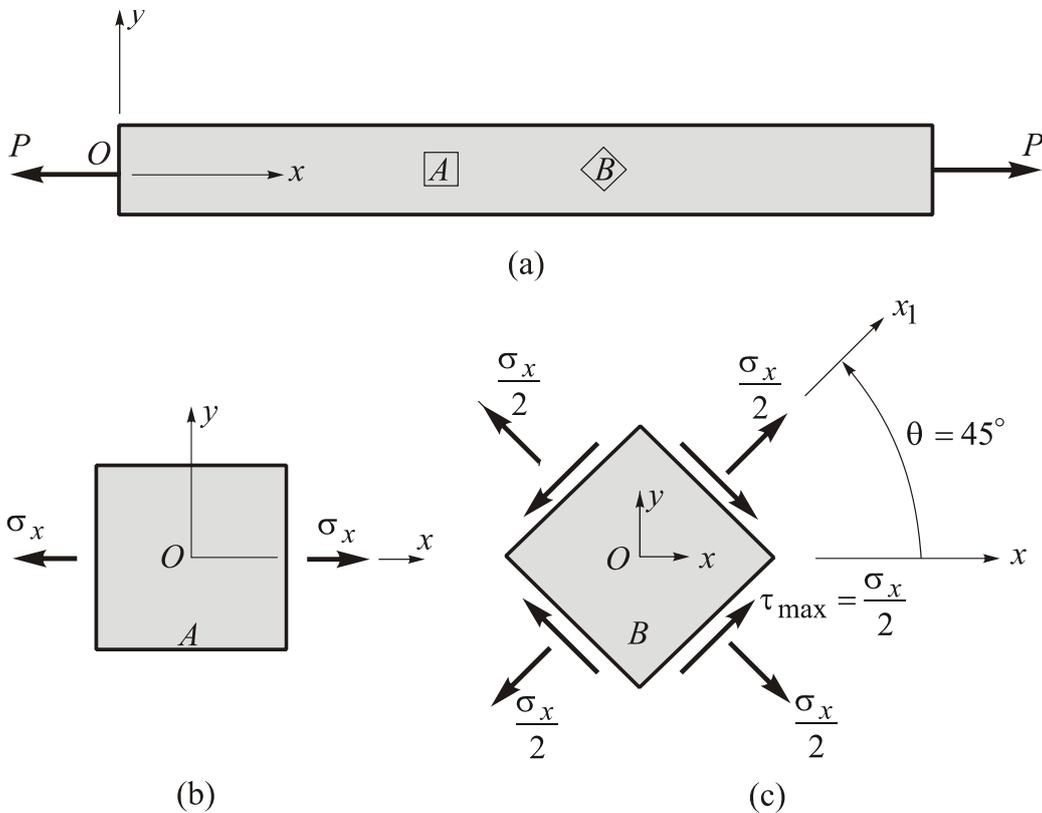


Fig. 2.14 Normal and shear stresses acting on stress elements oriented at  $\theta = 0^\circ$  and  $\theta = 45^\circ$  for a bar in tension

In the case of element  $B$  (Fig. 2.14c), *both normal and shear stresses act on all faces*. Consider, for instance, the face at  $45^\circ$  (the upper right-hand face). On this face the normal and shear stresses (from Eqs. 2.17a and b) are  $+\sigma_x/2$  and  $-\sigma_x/2$  respectively. Hence, the normal stress is tension (positive) and the shear stress acts clockwise (negative) against the element. The stresses on the remaining faces are obtained in a similar manner by substituting and  $\theta = 135^\circ$ ,  $-45^\circ$ , and  $-135^\circ$  into Eqs. (2.17a and b). Thus, in this special case of an element oriented at  $\theta = 45^\circ$ , the normal stresses on all four faces are the same (equal to  $\sigma_x/2$ ) and all four shear stresses have the maximum magnitude (equal to  $\sigma_x/2$ ).

**Note. The shear stresses acting on perpendicular planes are equal in magnitude and have directions either toward, or away from, the line of intersection of the planes (common edge).**

If a bar is loaded in compression instead of tension, the stress  $\sigma_x$  will be compressive and will have a negative value. Consequently, all stresses acting on stress elements will have directions opposite to those for a bar in tension. Eqs. (2.17a and b) can still be used for the calculations by substituting  $\sigma_x$  as a negative quantity.

Even though the maximum shear stress in an axially loaded bar is only one-half the maximum normal stress, the *shear stress may cause failure if the material is much weaker in shear than in tension* (it depends of, for example, internal crystalline structure of metals). An example of a shear failure is pictured in Fig. 2.15, which shows a block of wood that was loaded in compression and failed by shearing along a  $45^\circ$  plane. A similar type of behavior occurs in mild steel loaded in tension. During a tensile test of a flat bar of low-carbon steel with polished surfaces, visible *slip bands* appear on the sides of the bar at approximately  $45^\circ$  to the axis (Fig. 2.16). These bands indicate that the material is failing in shear along the planes on which the shear stress is maximum. Such bands were first observed by G. Piobert in 1842 and W. Luders in 1860, and today they are called either *Luders' bands* or *Piobert's bands*. They begin to appear when the yield stress is reached in the bar (point  $B$  in Fig. 2.1).

In result, the state of stress described in this chapter is called *uniaxial stress*, for the obvious reason that the bar is subjected to simple tension or compression in just one direction and  $\sigma_x$  in  $\theta = 0$  is called *principal stress*. The most important orientations of stress elements for *uniaxial stress state* are  $\theta = 0$  and  $\theta = 45^\circ$  (Fig. 2.14); the former has the maximum normal stress and the latter has the maximum shear stress. If sections are cut through the bar at other angles, the stresses acting on the faces of the corresponding stress elements can be determined from Eqs. (2.17a and b). *Uniaxial stress state is a special case of a more general stress state known as plane stress.*

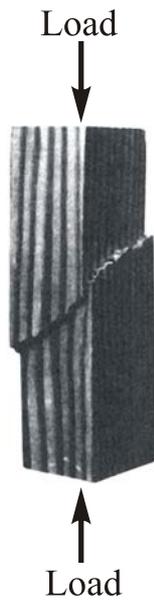


Fig. 2.15 Shear failure 45° plane of a wood block loaded in compression



Fig. 2.16 Slip bands (or a Luders' bands) in a polished steel specimen loaded in tension



## EXAMPLES

### Example 2.3

A prismatic bar having cross-sectional area  $A = 1200 \text{ mm}^2$  is compressed by an axial load  $P = 90 \text{ kN}$  (see figure (a)). (1) Determine the stresses acting on an inclined section  $p-q$  cut through the bar at an angle  $\theta = 25^\circ$ . (2) Determine the complete state of stress for  $\theta = 25^\circ$  and show the stresses on a properly oriented stress element.

**Solution** (1) *Calculation of stresses acting on an inclined section.* To find the stresses acting on a section at  $\theta = +25^\circ$ , we first calculate the normal stress  $\sigma_x$ , acting on a cross section:

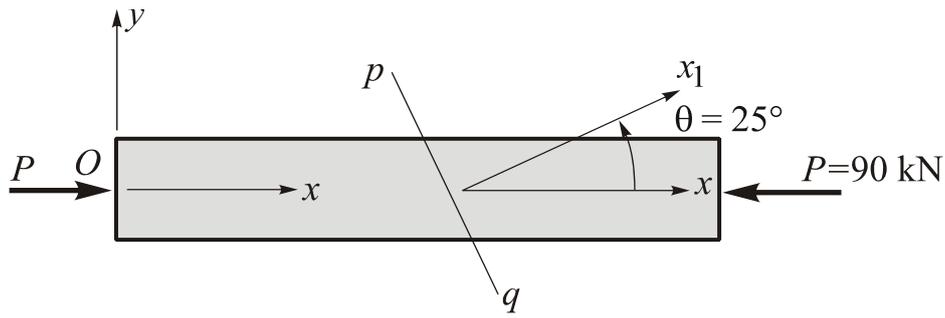
$$\sigma_x = -\frac{P}{A} = -\frac{90 \text{ kN}}{1200 \text{ mm}^2} = -75 \text{ MPa},$$

where the minus sign indicates that the stress is compressive. Next, we calculate the normal and shear stresses from Eqs. (2.17a and b) with  $\theta = +25^\circ$ , as follows:

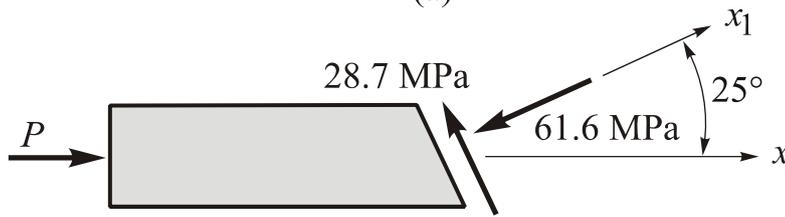
$$\sigma_\theta = \sigma_x \cos^2 \theta = (-75 \text{ MPa})(\cos 25^\circ)^2 = -61.6 \text{ MPa},$$

$$\tau_\theta = -\sigma_x \sin \theta \cos \theta = -(-75 \text{ MPa})(\sin 25^\circ)(\cos 25^\circ) = +28.7 \text{ MPa}.$$

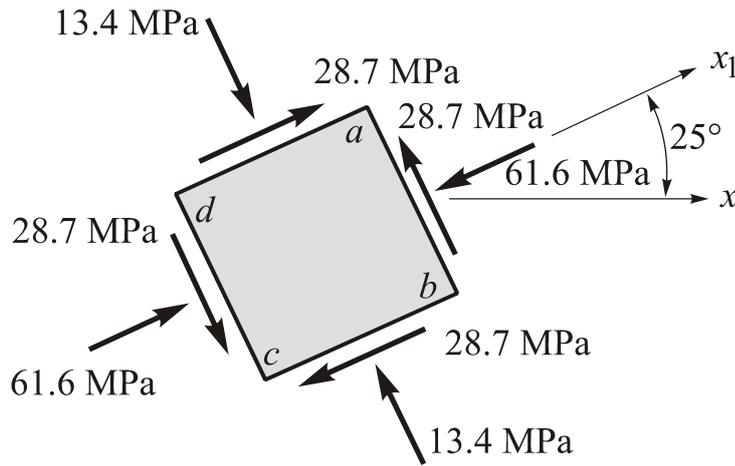
These stresses are shown acting on the inclined section in the figure (b). The normal stress  $\sigma_\theta$  is negative (compressive). The shear stress  $\tau_\theta$  is positive (and counterclockwise).



(a)



(b)



(c)

(2) *Complete state of stress for  $\theta = 25^\circ$ .* To determine the complete state of stress, we need to find the stresses acting on all faces of a stress element oriented at  $+25^\circ$  (see figure (c)). Face *ab*, for which  $\theta = +25^\circ$ , has the same orientation as the inclined plane shown in the figure (b). Therefore, the stresses are the same as those given above.

The stresses on the opposite face *cd* are the same as those on face *ab*, which can be verified by substituting  $\theta = 25^\circ + 180^\circ = 205^\circ$  into Eqs. (2.17a and b).

For face *bc* we substitute  $\theta = 25^\circ - 90^\circ = -65^\circ$  into Eqs. (2.17a and b) and obtain

$$\sigma_\theta = -13.4 \text{ MPa (compressive)}, \tau_\theta = -28.7 \text{ MPa (clockwise)}.$$

These same stresses apply to the opposite face  $ad$ , as can be verified by substituting  $\theta = 25^\circ + 90^\circ = 115^\circ$  into Eqs. (2.17a and b). The normal stress is compressive and the shear stress is clockwise.

The complete state of stress is shown by the stress element (see figure (c)). A sketch of this kind is the way to show the directions of the stresses and the orientations of the planes on which they act.

### Example 2.4

A compressed bar having a square cross of width  $b$  must support a load  $P = 37 \text{ kN}$  (see figure (a)). The two parts of the bar are connected by a glued joint along plane  $p-q$ , which is at an angle  $\alpha = 40^\circ$  to the vertical. The bar is constructed of a structural plastic for which the allowable stresses in compression and shear are  $7.6 \text{ MPa}$  and  $4.1 \text{ MPa}$ , respectively. In addition, the allowable stresses in the glued joint are  $5.2 \text{ MPa}$  in compression and  $3.4 \text{ MPa}$  in shear. Determine the minimum width  $b$  of the bar.

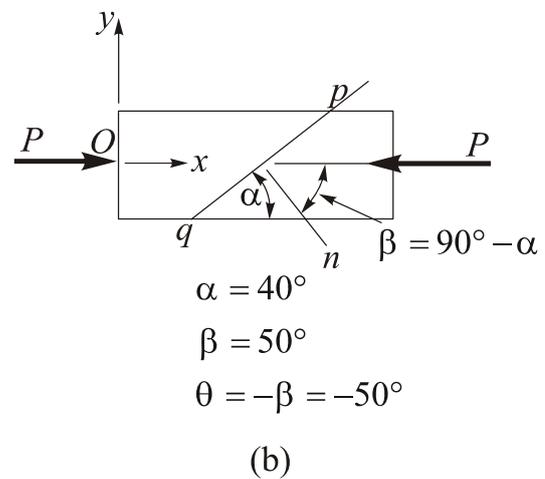
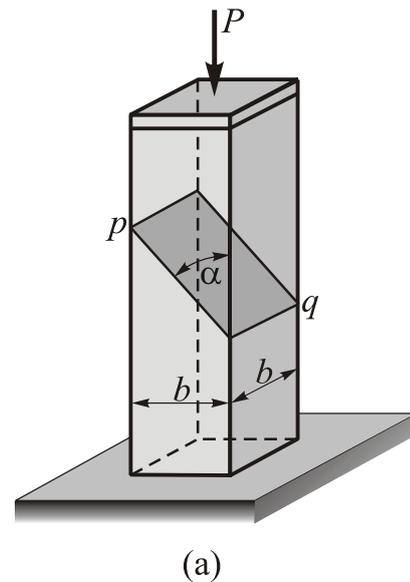
**Solution** The sketch of the bar is represented in the figure (b) in horizontal position similar to considered above, to use the equations for the stresses on an inclined section similar to Figs. 2.11 and 2.12.

With the bar in this position, we see that the normal  $n$  to the plane of the glued joint (plane  $p-q$ ) makes an angle  $\beta = 90^\circ - \alpha$ , or  $50^\circ$ , with the axis of the bar. Since the angle  $\theta$  is defined as positive when counterclockwise (Fig. 2.12), we believe that  $\theta = -50^\circ$  for the glued joint.

The cross-sectional area of the bar is related to the load  $P$  and the stress  $\sigma_x$  on the cross sections by the equation

$$A = \frac{P}{\sigma_x}. \quad (\text{a})$$

Therefore, to find the required area, we first must determine the value of  $\sigma_x$  corresponding to each of the four allowable stresses. Then the smallest value of  $\sigma_x$  will



determine the required area. The values of  $\sigma_x$  are obtained by rearranging Eqs. (2.17a and b) as follows:

$$\sigma_x = \frac{\sigma_\theta}{\cos^2 \theta}, \quad \sigma_x = -\frac{\tau_\theta}{\sin \theta \cos \theta}. \quad (\text{b})$$

We will now apply these equations to the glued joint and to the plastic.

(1) *Values of  $\sigma_x$  based upon the allowable stresses in the glued joint.* For compression in the glued joint we have  $\sigma_\theta = \sigma_{\text{all}_c} = -5.2 \text{ MPa}$  and  $\theta = -50^\circ$ . Substituting into Eq. (b), we get

$$\sigma_x = \frac{-5.2 \text{ MPa}}{(\cos(-50^\circ))^2} = -12.59 \text{ MPa}. \quad (\text{c})$$

For shear in the glued joint we have an allowable stress  $\tau_{\text{all}} = 3.4 \text{ MPa}$ . However, it is not immediately evident whether  $\tau_\theta$  is  $+3.4 \text{ MPa}$  or  $-3.4 \text{ MPa}$ . *One approach* is to substitute both  $+3.4 \text{ MPa}$  and  $-3.4 \text{ MPa}$  into second Eq. (b) and then select the value of  $\sigma_x$  that is negative. The other value of  $\sigma_x$  will be positive (tension) and does not apply to this bar, which is in compression. *Second approach* is to inspect the bar itself (see figure (b)) and observe from the directions of the loads that the shear stress will act clockwise against plane  $p-q$ , which means that the shear stress is negative. Therefore, we substitute  $\tau_\theta = -3.4 \text{ MPa}$  and  $\theta = -50^\circ$  second Eq. (b) and obtain

$$\sigma_x = -\frac{-3.4 \text{ MPa}}{(\sin(-50^\circ))(\cos(-50^\circ))} = -6.9 \text{ MPa}. \quad (\text{d})$$

(2) *Values of  $\sigma_x$  based upon the allowable stresses in the plastic.* The maximum compressive stress in the plastic occurs on a cross section. Therefore, since the allowable stress in compression  $\sigma_{\text{all}_c} = -7.6 \text{ MPa}$ , we know immediately that  $\sigma_x = -7.6 \text{ MPa}$ .

The maximum shear stress occurs on a plane at  $45^\circ$  and is numerically equal to  $\sigma_x / 2$  (see Eq. 2.19). Since the allowable stress in shear  $\tau_{\text{all}} = 4.1 \text{ MPa}$ , we obtain

$$\sigma_x = -2.05 \text{ MPa}. \quad (\text{e})$$

The same result can be obtained from second Eq. (b) by substituting  $\tau_\theta = 4.1 \text{ MPa}$  and  $\theta = 45^\circ$ .

(3) *Minimum width of the bar.* Comparing the four values of  $\sigma_x$ , we see that the smallest is  $\sigma_x = -6.9 \text{ MPa}$ . Therefore, this value we will take into

account in design. Substituting into Eq. (a), and using only numerical values, we obtain the required area:

$$A = \frac{37,000 \text{ N}}{6.9 \times 10^6 \text{ N/m}^2} = 5.36 \times 10^{-3} \text{ m}^2.$$

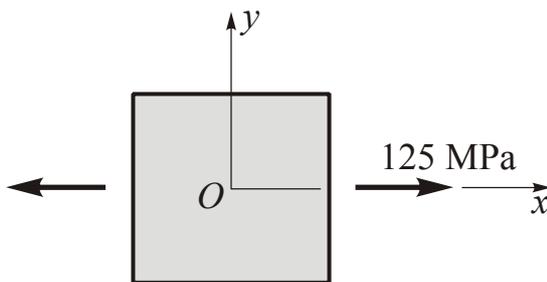
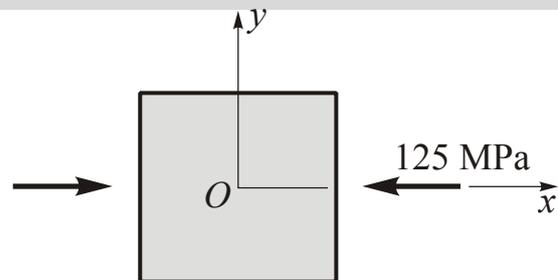
Since the bar has a square cross section ( $A = b^2$ ), the minimum width is

$$b_{\min} = \sqrt{A} = \sqrt{5.36 \times 10^{-3}} = 7.32 \text{ cm}.$$



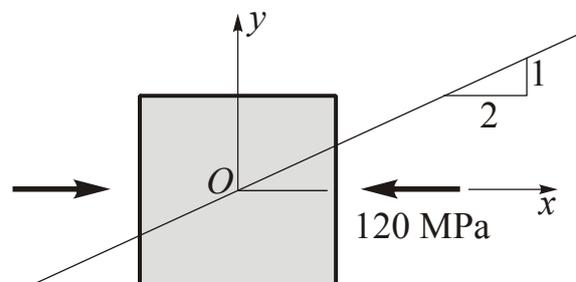
## PROBLEMS

**Problem 2.37** An element in uniaxial stress is subjected to tensile stresses  $\sigma_x = 125 \text{ MPa}$ , as shown in the figure. Determine: (1) the stresses acting on an element oriented at an angle  $\theta = 21.8^\circ$  from the  $x$  axis, and (2) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.



**Problem 2.38** An element in uniaxial stress is subjected to compressible stresses  $\sigma_x = 125 \text{ MPa}$ , as shown in the figure. Determine: (1) the stresses acting on an element oriented at an angle  $\theta = 21.8^\circ$  from the  $x$  axis, and (2) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.

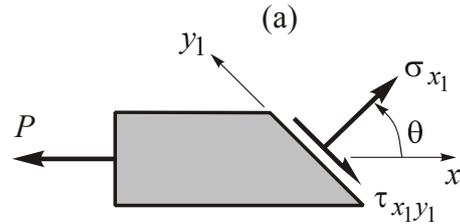
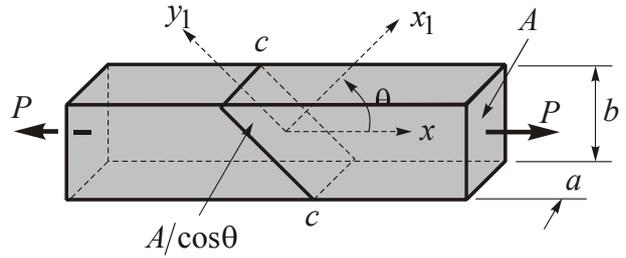
**Problem 2.39** An element in uniaxial stress is subjected to a compressive stress of  $120 \text{ MPa}$ , as shown in the figure. Determine: (1) the stresses acting on an element oriented at a slope of 1 on 3 (see figure), and (2) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.



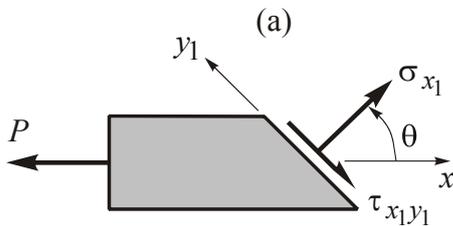
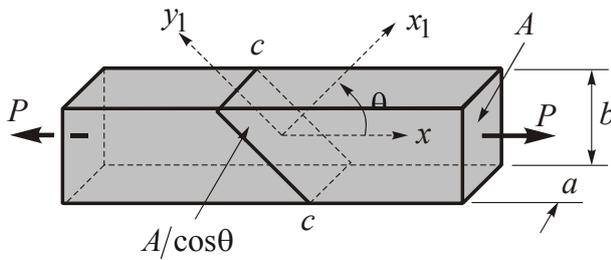
**Problem 2.40** What are the maximum normal and shearing stresses

in a circular bar of 30 mm diameter subjected to an axial compression load of  $F = 90$  kN?

**Problem 2.41** Determine the maximum axial load  $P$  that can be applied to the rectangular wooden bar of figure (a) without exceeding a shearing stress of  $\tau_{x_1y_1} = 11$  MPa or a normal stress of  $\sigma_{x_1} = 30$  MPa on the inclined plane  $c-c$  parallel to its grain. Use  $\theta = 60^\circ$ ,  $a = 20$  mm, and  $b = 50$  mm.



(b)



(b)

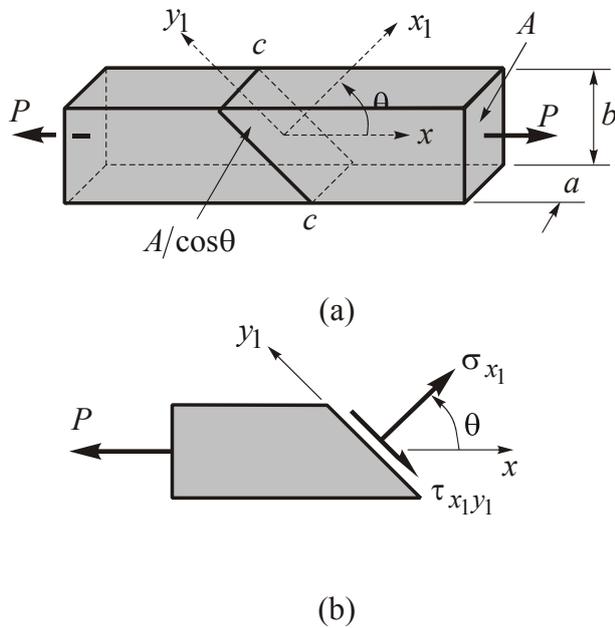
**Problem 2.42** Determine the maximum axial load  $P$  that can be applied to the rectangular wooden bar of figure (a) without exceeding a shearing stress of  $\tau_{x_1y_1} = 5.5$  MPa or a normal stress of  $\sigma_{x_1} = 15$  MPa on the inclined plane  $c-c$  parallel to its grain. Use  $\theta = 60^\circ$ ,  $a = 20$  mm, and  $b = 50$  mm.

**Problem 2.43** A cylinder of 50-mm inner radius and 5-mm wall thickness has a welded spiral seam at an angle of  $40^\circ$  with the axial ( $x$ ) direction. The cylinder is subjected to an axial compressive load of 10 kN applied through rigid end plates. Determine the normal  $\sigma$  and shear  $\tau$  stresses acting simultaneously in the plane of welding.

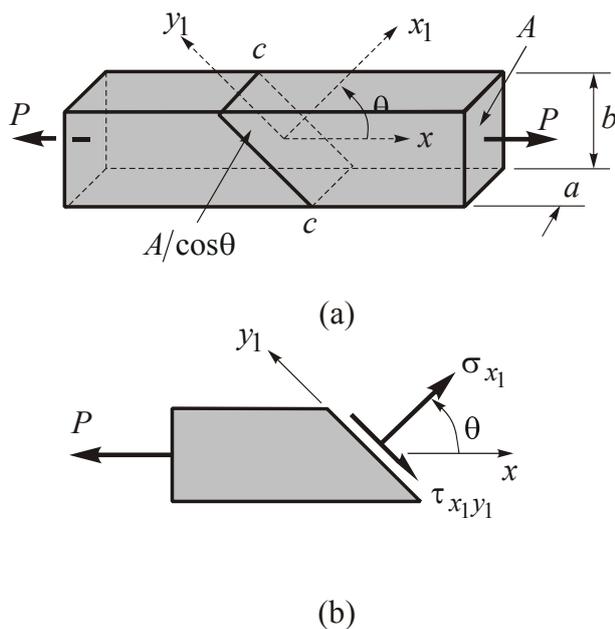
**Problem 2.44** A cylinder of 50-mm inner radius and 2.5-mm wall thickness has a welded spiral seam at an angle of  $60^\circ$  with the axial ( $x$ ) direction. The cylinder is subjected to an axial compressed load of 10 kN applied through rigid end plates. Determine the normal  $\sigma$  and shear  $\tau$  stresses acting simultaneously in the plane of welding.

**Problem 2.45** Calculate the normal and shearing stresses on a plane through the bar of the figure (a) that makes an angle of  $30^\circ$  with the

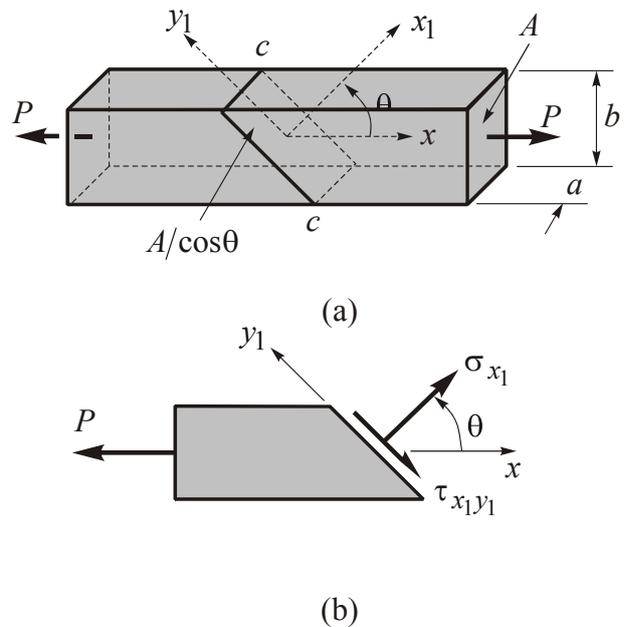
direction of force  $P = 120$  kN. Use  $a = 15$  mm and  $b = 30$  mm.



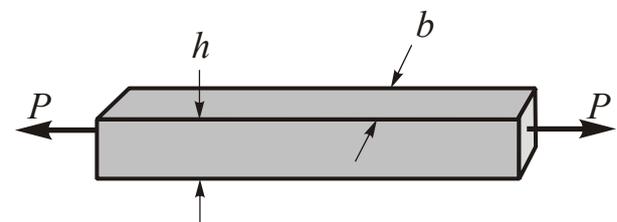
**Problem 2.46** The stresses on an inclined plane  $c-c$  in a bar in tension (see figure (a)) are  $\sigma_{x_1} = 20$  MPa and  $\tau_{x_1y_1} = 10$  MPa. Determine the axial load  $P$  and the angle  $\theta$  for  $a = 30$  mm and  $b = 60$  mm.



**Problem 2.47** Calculate the normal and shearing stresses on a plane through the bar of figure (a) that makes an angle of  $15^\circ$  with the direction of force  $P = 120$  kN. Use  $a = 15$  mm and  $b = 30$  mm.

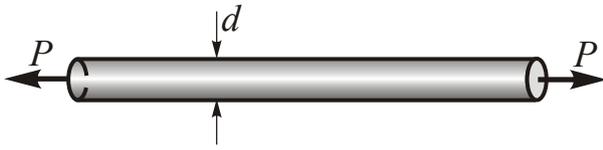


**Problem 2.48** A steel bar of square cross section ( $35$  mm  $\times$   $35$  mm) carries a tensile load  $P$  (see figure). The allowable stresses in tension and shear are  $125$  MPa and  $75$  MPa, respectively. Determine the maximum permissible load  $P_{\max}$ .

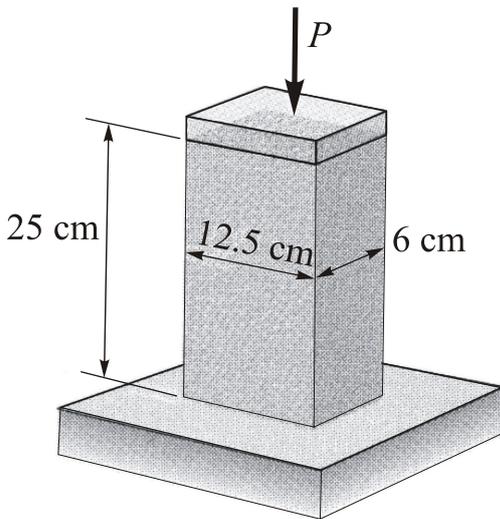


**Problem 2.49** A circular steel rod of diameter  $d$  is subjected to a tensile force  $P = 80$  kN (see figure). The allowable stresses in tension and

shear are 120 MPa and 55 MPa, respectively. What is the minimum permissible diameter  $d_{\min}$  of the rod?

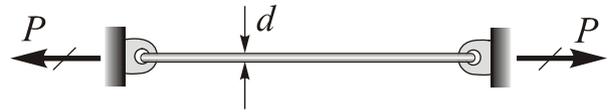


**Problem 2.50** A standard brick having dimensions  $25\text{ cm} \times 12.5\text{ cm} \times 6\text{ cm}$  is compressed lengthwise in a testing machine (see figure). If the ultimate shear stress for brick is 7.5 MPa and the ultimate compressive stress is 25 MPa, what force  $P_{\max}$  is required to break the brick?

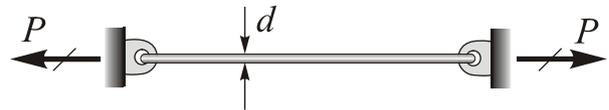


**Problem 2.51** A brass wire of diameter  $d = 2\text{ mm}$  is stretched between rigid supports so that the tensile force is  $P = 140\text{ N}$  (see figure). What is the maximum permissible temperature drop  $\Delta T$  if the allowable shear stress in the wire is 80 MPa? (The coefficient of

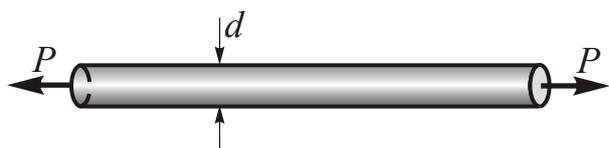
thermal expansion for the wire is  $19.5 \times 10^{-6}/^\circ\text{C}$  and the modulus of elasticity is 105 GPa).



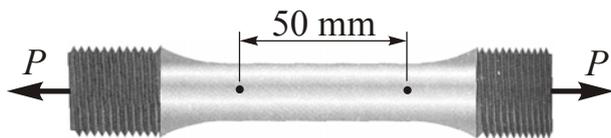
**Problem 2.52** A steel wire of diameter  $d = 3\text{ mm}$  is stretched between rigid supports with an initial tension  $P$  of 200 N (see figure). (1) If the temperature is then lowered by  $50^\circ\text{F}$ , what is the maximum shear stress  $\tau_{\max}$  in the wire? (2) If the allowable shear stress is 125 MPa, what is the maximum permissible temperature drop? (Assume that the coefficient of thermal expansion is  $6.5 \times 10^{-6}/^\circ\text{F}$  and the modulus of elasticity is 205 MPa).



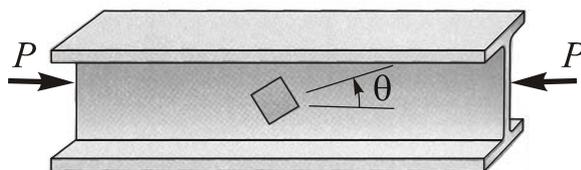
**Problem 2.53** A 20-mm diameter steel bar is subjected to a tensile load  $P = 25\text{ kN}$  (see figure). (1) What is the maximum normal stress  $\sigma_{\max}$  in the bar? (2) What is the maximum shear stress  $\tau_{\max}$ ? (3) Draw a stress element oriented at  $45^\circ$  to the axis of the bar and show all stresses acting on the faces of this element.



**Problem 2.54** During a tension test of a mild-steel specimen (see figure), the extensometer shows an elongation of 0.04 mm over a gage length of 50 mm. Assume that the steel is stressed below the proportional limit and that the modulus of elasticity  $E = 205 \text{ MPa}$ . (1) What is the maximum normal stress  $\sigma_{\max}$  in the specimen? (2) What is the maximum shear stress  $\tau_{\max}$ ? (3) Draw a stress element oriented at an angle of  $45^\circ$  to the axis of the bar and show all stresses acting on the faces of this element.

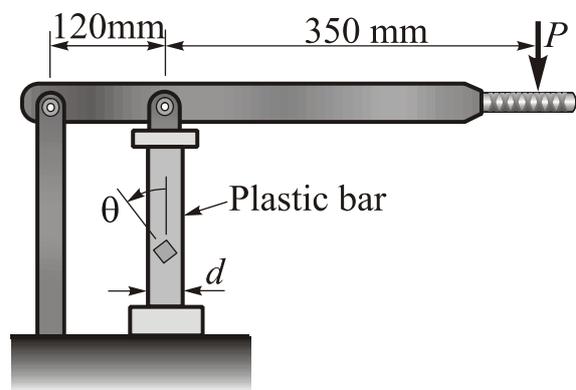


**Problem 2.55** A compressed member in a truss is fabricated from a wide-flange steel section (see figure). The cross-sectional area  $A = 6640 \text{ mm}^2$  and the axial load  $P = 410 \text{ kN}$ . Determine the normal and shear stresses acting on all faces of stress elements located in the web of the beam and oriented at: (1) an angle  $\theta = 0^\circ$ , (2) an angle  $\theta = 22.5^\circ$ , and (3) an angle  $\theta = 45^\circ$ . In each case, show the stresses on a sketch of a properly oriented element.

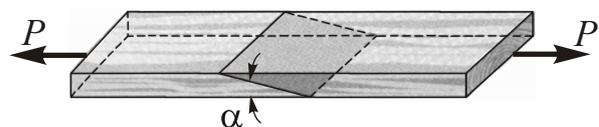


**Problem 2.56** A plastic bar of diameter  $d = 25 \text{ mm}$  is compressed in a

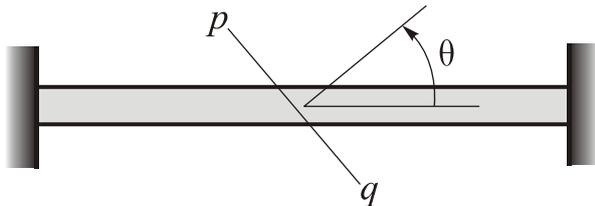
testing device by a force  $P = 110 \text{ N}$  applied as shown in the figure. Determine the normal and shear stresses acting on all faces of stress elements oriented at: (1) an angle  $\theta = 0^\circ$ , (2) an angle  $\theta = 22.5^\circ$ , and (3) an angle  $\theta = 45^\circ$ . In each case, show the stresses on a sketch of a properly oriented element.



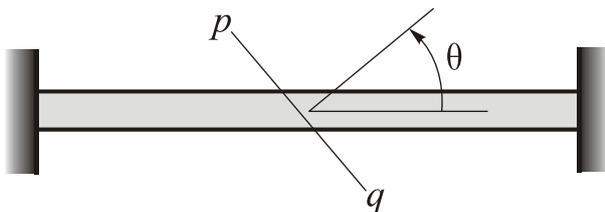
**Problem 2.57** Two boards are joined by gluing along a scarf joint as shown in the figure. For practical reasons, the angle  $\alpha$  between the plane of the joint and the faces of the boards must be  $45^\circ$  or less. Under a tensile load  $P$ , the normal stress in the boards is  $5.0 \text{ MPa}$ . (1) What are the normal and shear stresses acting on the glued joint if  $\alpha = 20^\circ$ ? (2) If the allowable shear stress on the joint is  $2.25 \text{ MPa}$ , what is the largest permissible value of the angle  $\alpha$ ? (3) For what angle  $\alpha$  will the shear stress on the glued joint be numerically equal to twice the normal stress on the joint?



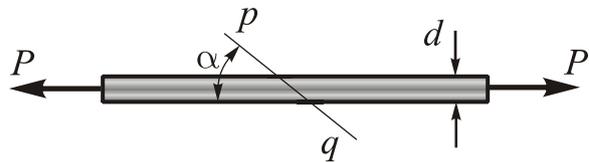
**Problem 2.58** A copper bar is held closely (but without any initial stress) between rigid supports (see figure). The allowable stresses on the inclined plane  $p-q$ , for which  $\theta = 53.13^\circ$ , are specified as 50 MPa in compression and 30 MPa in shear. (1) What is the maximum permissible temperature rise  $\Delta T$  if the allowable stresses on plane  $p-q$  are not to be exceeded? (2) If the temperature increases by the maximum permissible amount, what are the stresses on plane  $p-q$ ? Assume  $\alpha = 9.4 \times 10^{-6} / ^\circ\text{F}$  and  $E = 125 \text{ GPa}$ .



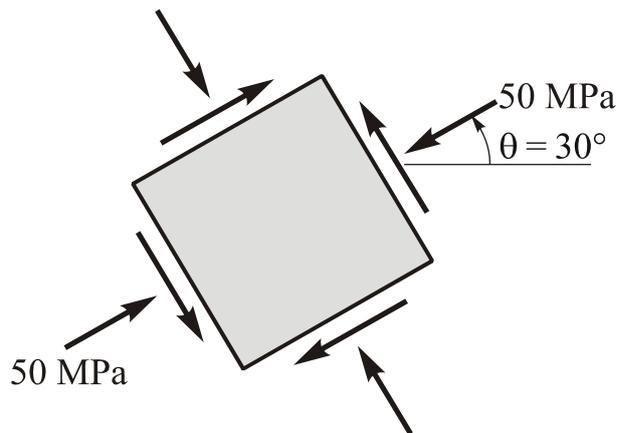
**Problem 2.59** A plastic bar is held closely between rigid supports at room temperature ( $20^\circ\text{C}$ ) but with no initial stress (see figure). When the temperature of the bar is raised to  $70^\circ\text{C}$ , the compressive stress on an inclined plane  $p-q$  becomes 12.0 MPa. (1) What is the shear stress on plane  $p-q$ ? (2) Draw a stress element oriented to plane  $p-q$  and show the stresses acting on all faces of this element. Assume  $\alpha = 100 \times 10^{-6} / ^\circ\text{F}$  and  $E = 3.0 \text{ GPa}$ .



**Problem 2.60** A circular brass bar of diameter  $d$  is composed of two segments brazed together on a plane  $p-q$  making an angle  $\alpha = 36^\circ$  with the axis of the bar (see figure). The allowable stresses in the brass are 90 MPa in tension and 50 MPa in shear. On the brazed joint, the allowable stresses are 40 MPa in tension and 24 MPa in shear. If the bar must resist a tensile force 30 kN, what is the minimum required diameter  $d_{\text{min}}$  of the bar?

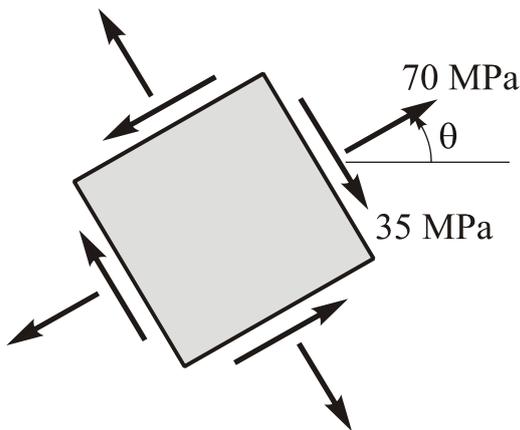


**Problem 2.61** A prismatic bar is subjected to an axial force that produces a compressive stress of  $\sigma = 50 \text{ MPa}$  on a plane at an angle  $\theta = 30^\circ$  (see figure). Determine the stresses acting on all faces of a stress element oriented at  $\theta = 60^\circ$  and show the stresses on a sketch of the element.

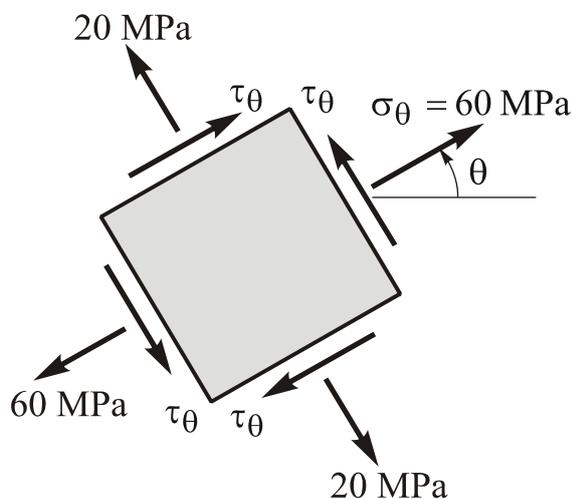


**Problem 2.62** A prismatic bar is subjected to an axial force that produces

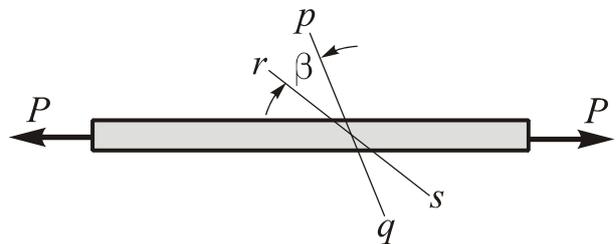
a tensile stress  $\sigma_\theta = 70$  MPa and a shear stress  $\tau_\theta = 35$  MPa on a certain inclined plane (see figure). Determine the stresses acting on all faces of a stress element oriented at  $\theta = 50^\circ$  and show the stresses on a sketch of the element.



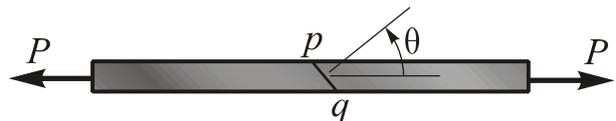
**Problem 2.63** Acting on the sides of a stress element cut from a bar in uniaxial stress are normal stresses of 60 MPa and 20 MPa, as shown in the figure. (1) Determine the angle  $\theta$  and the shear stress  $\tau_\theta$ . (2) What are the maximum normal stress  $\sigma_{\max}$  and the maximum shear stress  $\tau_{\max}$  in the material?



**Problem 2.64** The normal stress on plane  $p-q$  of a prismatic bar in tension (see figure) is found to be 56 MPa. On plane  $r-s$ , which makes an angle  $\beta = 30^\circ$  with plane  $p-q$ , the stress is found to be 22.6 MPa. Determine the maximum normal stress  $\sigma_{\max}$  and maximum shear stress  $\tau_{\max}$  in the bar.



**Problem 2.65** A tensiled member is to be made of two pieces of plastic glued along plane  $p-q$  (see figure). For practical reasons, the angle  $\theta$  must be between  $25^\circ$  and  $45^\circ$ . The allowable stresses on the glued joint in tension and shear are 14 MPa and 9 MPa, respectively. (1) Determine the angle  $\theta$  so that the bar will carry the largest load  $P$ . (2) Determine the maximum allowable load  $P_{\max}$  if the cross-sectional area of the bar is  $600 \text{ mm}^2$ . Assume that the strength of the glued joint controls the design.



## 2.3 Strain-Energy Density and Strain Energy in Uniaxial Stress State

### 2.3.1 Strain-Energy Density

*Strain energy* is one of fundamental concepts in mechanics of materials. The work done by external forces in producing deformation is stored within the body as strain energy. *For a perfectly elastic body, no dissipation of energy occurs*, and all the stored energy is recoverable upon unloading. The concept of strain energy is useful as applied to the solution of problems involving both static and dynamic loads. In this section we introduce the subject of strain energy, stored in axially loaded members subjected to static loads, beginning from the its elementary part, stored in an infinitesimal element (volume) of elastic material  $dV$ . **Note. It is assumed that inside this element strain energy is distributed uniformly.**

Let us express the strain energy owing to uniaxial stress state by considering an element subjected to a slowly increasing normal stress  $\sigma_x$  (Fig. 2.17a). The element is assumed to be initially free of stress. The force acting on each  $x$  face,  $\sigma_x dydz$ , elongates the element an amount  $\varepsilon_x dx$  where  $\varepsilon_x$  is the  $x$ -directed strain. In the case of a linearly elastic material,  $\sigma_x = E\varepsilon_x$ . The average force acting on the elastic element during the straining is  $\frac{1}{2}\sigma_x dydz$ . Thus the strain energy  $U$  corresponding to the work done by this force,  $\frac{1}{2}\sigma_x dydz \varepsilon_x dx$ , is expressed as

$$dU = \frac{1}{2}\sigma_x \varepsilon_x (dx dy dz) = \frac{1}{2}\sigma_x \varepsilon_x dV, \quad (2.20)$$

where  $dV$  is the volume of the element. The unit of work and energy in SI is the *joule* (J), equal to a newton-meter (N·m).

The strain energy per unit volume,  $dU/dV$ , is referred to as the *strain-energy density*, designated  $U_0$ . From the foregoing, we express it in two forms:

$$U_0 = \frac{1}{2}\sigma_x \varepsilon_x = \frac{\sigma_x^2}{2E} = \frac{E\varepsilon_x^2}{2}. \quad (2.21)$$

These equations give the strain-energy density in a linearly elastic material in terms of the normal stress  $\sigma$  and the normal strain  $\varepsilon$ .

The expressions in Eqs. (2.21) have a simple geometric interpretation. They are equal to the area  $\sigma\varepsilon/2$  of the *triangle below the stress-strain diagram* for a material that follows Hooke's law ( $\sigma = E\varepsilon$ ) (see Figs. 2.17 b, 2.18 a).

In more general situations where the material does not follow Hooke's law the strain-energy density is still equal to the area below the stress-strain curve, but the area must be evaluated in each particular case (see dashed line on Fig. 2.17 b).

The area above the stress-strain curve is known as the *complementary energy density*, denoted  $U_0^*$  as seen in the Fig. 2.17 b.

In SI units the strain-energy density is expressed in joules per cubic meter ( $\text{J/m}^3$ ) or in *pascals*. As the stress ( $\sigma_x$ ) is squared, the strain energy is always a positive quantity, and Eq. (2.21) applies for a member in tension or compression.

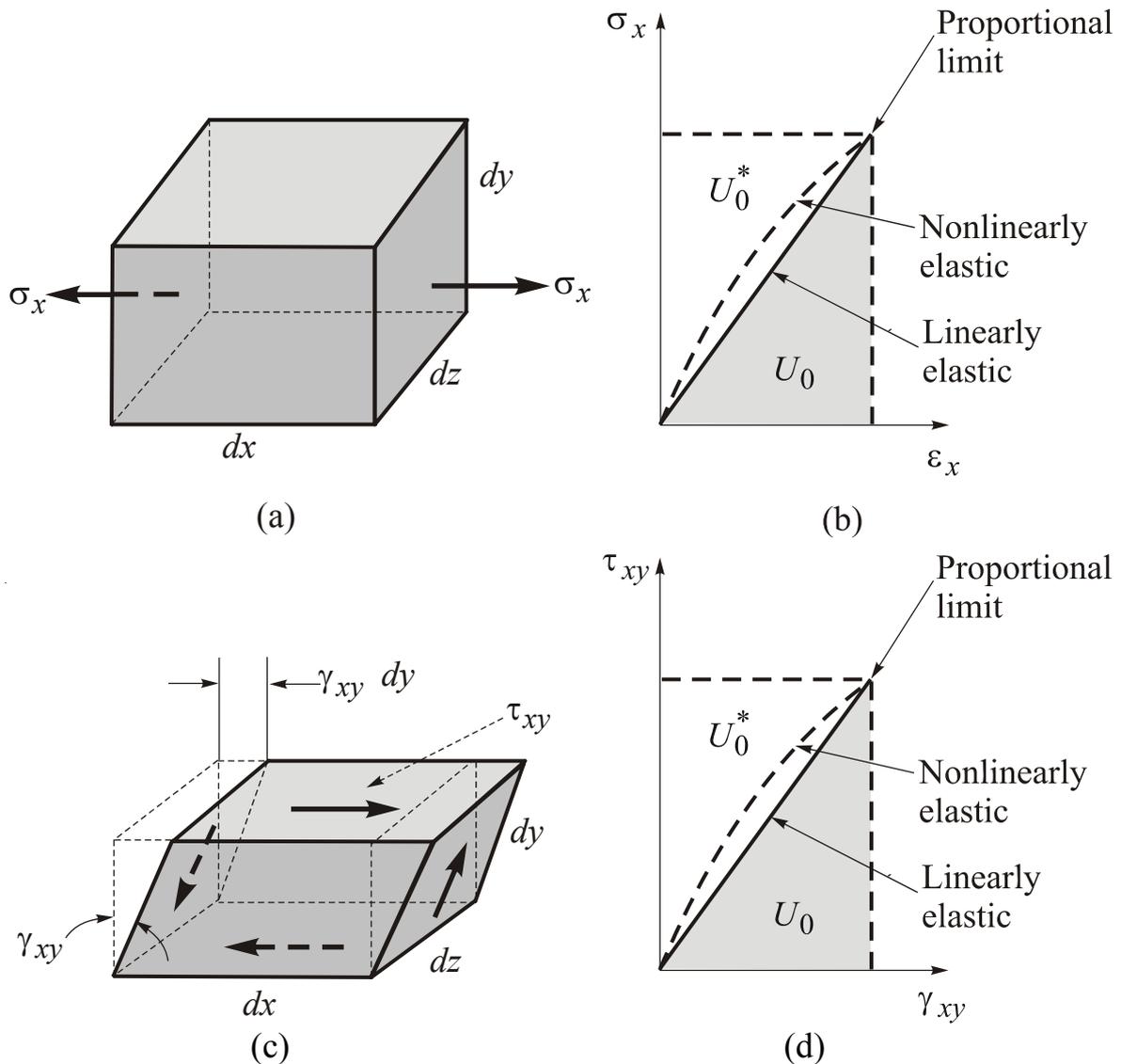


Fig. 2.17 To strain-energy density calculation: (a) an element in tension; (b) stress-strain diagram in tension; (c) an element in shear; (d) stress-strain diagram in shear

The strain-energy density when the material is stressed to the yield strength is called the *modulus of resilience*. It is equal to the area under the straight-line portion of the stress-strain diagram (Fig. 2.18a) and measures the ability of the material to absorb energy without permanent deformation:

$$U_r = \frac{\sigma_{pr}^2}{2E}. \quad (2.22)$$

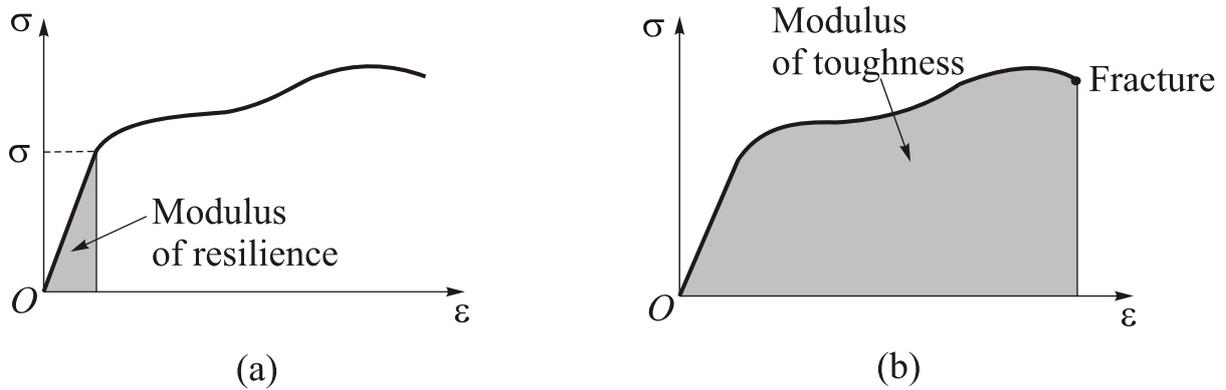


Fig. 2.18 Typical stress-strain diagram: (a) modulus of resilience and (b) modulus of toughness

For example, the value of the modulus of resilience for a mild steel is  $\sigma_{pr}^2 / 2E = (250 \times 10^6)^2 / 2(200 \times 10^9) = 156 \text{ kJ/m}^3$ . Another important quantity is known as the *modulus of toughness*. It is equal to the area under a complete stress-strain curve (Fig. 2.18b). *Toughness is a measure of the material's ability to absorb energy without fracturing.* Clearly, it is related to the ductility as well as to the ultimate strength of a material.

The strain energy for uniaxial normal stress is obtained by integrating the strain-energy density, Eq. (2.21), over the volume of the member:

$$U = \int_{(V)} \frac{\sigma_x^2}{2E} dV. \tag{2.23}$$

The foregoing can be used for axial loading and bending of beams, because in bending all fibers are tensiled or compressed.

Now consider the element under the action of shearing stress  $\tau_{xy}$  (Fig. 2.17 c). From the figure we observe that force  $\tau_{xy} dx dz$  causes a displacement  $\gamma_{xy} dy$ . The stress varies linearly from zero to its final value as before, and therefore the average force equals  $\frac{1}{2} \tau_{xy} dx dz$ . The strain-energy density in pure shear is then

$$U_0 = \frac{1}{2} \tau_{xy} \gamma_{xy} = \frac{\tau_{xy}^2}{2G}. \tag{2.24}$$

The expressions in Eq. (2.24) have the same geometric interpretation as (2.21). They are equal to the area  $\tau_{xy} \gamma_{xy} / 2$  of the triangle below the stress-strain diagram for a material that follows Hooke's law ( $\tau = G\gamma$ ) (see Fig. 2.17 d).

The strain energy for shear stress is expressed as

$$U = \int_{(V)} \frac{\tau_{xy}^2}{2G} dV. \quad (2.25)$$

The integration is over the volume of the member. Equation (2.25) can be used for bars in torsion and transverse shear in beams.

The integration of Eq. 2.23 is the most simple in the case of a prismatic bar of length  $L$  subjected to a tensile force  $P$  (Fig. 2.19), if we assume that the stresses and strains are distributed over the cross-section uniformly (well-known *hypothesis of plane sections*). Because the load is applied slowly, so that it gradually increases from zero to its maximum value  $P$ , such a load is called a *static load* (i. e. there are no dynamic or inertial effects due to motion). The bar gradually elongates as the load is applied, eventually reaching its maximum elongation  $\delta$  at the same time that the load reaches its full value  $P$ . Thereafter, the load and elongation remain unchanged.

During the loading process, the load  $P$  moves slowly through the distance  $\delta$  and does a certain amount of *work*. To evaluate this work, we recall from elementary mechanics that a constant force does work equal to the product of the force and the distance through which it moves. However, in our case the force varies in magnitude from zero to its maximum value  $P$ . To find the work done by the load under these conditions, we need to know the manner in which the force varies. This information is supplied by a *load-displacement diagram*, such as the one plotted in Fig. 2.20. On this diagram the vertical axis represents the axial load and the horizontal axis represents the corresponding elongation of the bar.

**Note.** The shape of the curve depends upon the properties of the material.

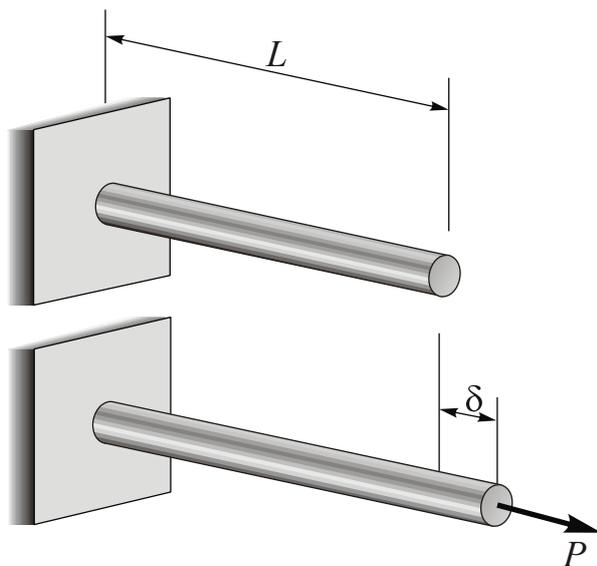


Fig. 2.19 Prismatic bar subjected to a statically applied load

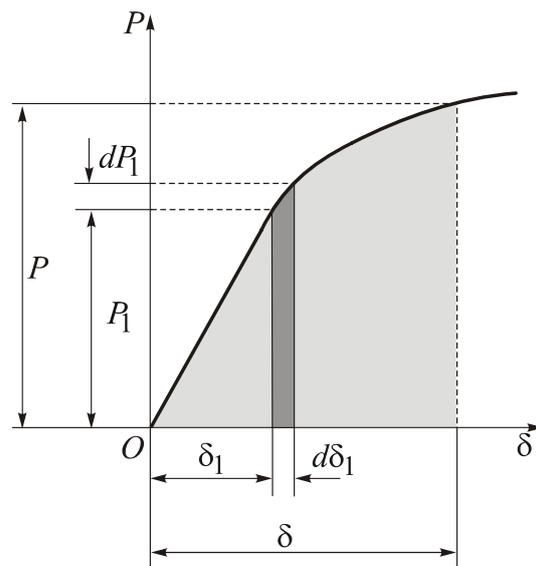


Fig. 2.20 Load-displacement diagram

Let us denote by  $P_1$  any value of the load between zero and the maximum value  $P$ , and the corresponding elongation of the bar by  $\delta_1$ . Then an increment  $dP_1$  in the load will produce an increment  $d\delta_1$  in the elongation. The work done by the load during this incremental elongation is the product of the load and the distance through which it moves, that is, the work equals  $P_1 d\delta_1$ . This work is represented in the figure by the area of the shaded rectangle below the load-displacement curve. The total work done by the load as it increases from zero to the maximum value  $P$  is the summation of all such elemental works:

$$W = \int_0^{\delta} P_1 d\delta_1 . \quad (2.26)$$

**Note, that the work done by the load is equal, in geometric sense, to the area below the load-displacement curve.**

When the load stretches the bar, strains are produced. The presence of these strains increases the energy level of the bar itself. The strain energy is the energy absorbed by the bar during the loading process. *From the principle of conservation of energy, we know that this strain energy is equal to the work done by the load* (we assume that no energy is added or subtracted in the form of heat). Therefore,

$$U = W = \int_0^{\delta} F_1 d\delta_1 , \quad (2.27)$$

in which  $U$  is the strain energy. Sometimes strain energy is referred to as *internal work* to distinguish it from the external work done by the load.

Work and energy are expressed in the same units. In SI, the unit of work and energy is the joule (J), which is equal to one newton meter ( $1 \text{ J} = 1 \text{ Nm}$ ).

In removing the force  $P$  (Fig. 2.19) from the bar, it will shorten. If the elastic limit of the material is not exceeded, the bar will return to its original position (length). If the limit is exceeded, a permanent set will remain. Thus, either all or part of the strain energy will be recovered in the form of work. This behavior is shown on the load-displacement diagram of Fig. 2.21. During loading, the work done by the load is equal to the area below the curve (area  $OABCDO$ ). When the load is removed, the load-displacement diagram follows line  $BD$  if point  $B$  is beyond the elastic limit and a permanent elongation  $OD$  remains. Thus, the strain energy recovered during unloading, called the *elastic strain energy*, is represented by the shaded triangle  $BCD$ . Area  $OABDO$  represents energy that is lost in the process of permanently deforming the bar. This energy is known as the *inelastic strain energy*.

Most structures are designed with the expectation that the material will remain within the elastic range under ordinary conditions of service. In the case of a bar in tension, the load at which the stress in the material reaches the elastic limit is represented by point  $A$  on the load-displacement curve (Fig. 2.21). As long as the load is below this value, all of the strain energy is recovered during unloading and no permanent elongation remains. Thus, the bar acts as an elastic spring, storing and releasing energy as the load is applied and removed.

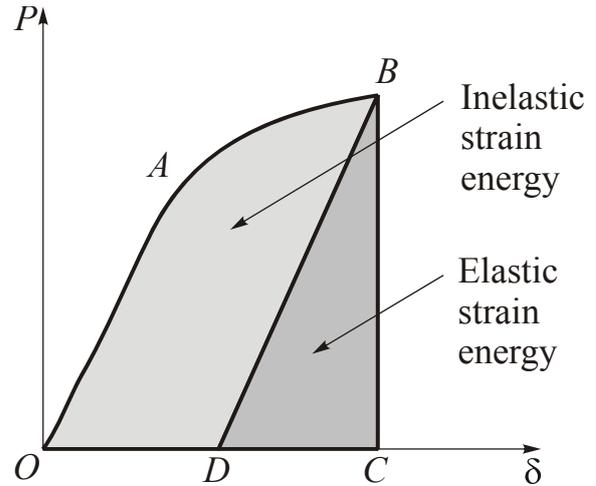


Fig. 2.21 Elastic and inelastic strain energy

Strain energy is a form of *potential energy* (or "*energy of position*") because it depends upon the relative locations of the particles or elements that make up the member. When a bar or a spring is compressed, its particles are crowded more closely together; when it is stretched, the distances between particles increase. In both cases the strain energy of the member increases as compared to its strain energy in the unloaded position.

Within the limitations of the assumption on uniform distribution of stresses over the cross section of tensiled or compressed bar, material of which follows Hooke's law (the load-displacement curve is a straight line (Fig. 2.22)), the strain energy  $U$  stored in the bar (equal to the work  $W$  done by the load) is

$$U = W = \frac{P\delta}{2}, \quad (2.28)$$

which is the area of the shaded triangle  $OAB$  in the figure.

**The principle that the work of the external loads is equal to the strain energy (for the case of linearly elastic behavior) was first stated by the French engineer B. P. E. Clapeyron (1799–1864) and is known as Clapeyron's theorem.**

After substitution into Hooke's law  $\sigma = P/A$  and  $\varepsilon = \Delta L/L$  the relationship between the load  $P$  and the elongation  $\Delta l = \delta$  for a linearly elastic material is given by the equation

$$\Delta l = \delta = \frac{PL}{EA}, \quad (2.29)$$

where  $L$  is the length of the bar,  $A$  and  $E$  are area of cross section and modulus of elasticity respectively.

Combining this equation with Eq. (2.28) allows us to express the strain energy of a linearly elastic bar in the following forms:

$$U = \frac{P^2 L}{2EA}, \quad \text{or} \quad U = \frac{EA\delta^2}{2L}. \quad (2.30 \text{ a, b})$$

The first equation expresses the strain energy as a function of the load and the second expresses it as a function of the elongation. From the first equation we see that increasing the length of a bar increases the amount of strain energy even though the load is unchanged (because more material is being strained by the load). On the other hand, increasing either the modulus of elasticity or the cross-sectional area decreases the strain energy because the strains in the bar are reduced.

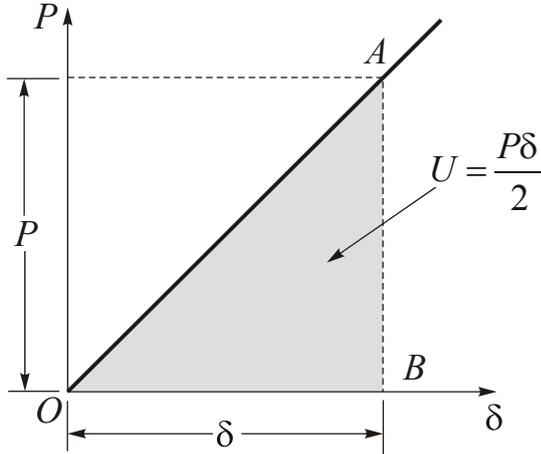


Fig. 2.22 Load-displacement diagram for a bar of linearly elastic material

The total strain energy  $U$  of a bar consisting of several segments is equal to the sum of the strain energies of the individual segments. For instance, the strain energy of the bar pictured in Fig. 2.23 equals the strain energy of segment  $AB$  plus the strain energy of segment  $BC$ . This concept is expressed in general terms by the following equation

$$U = \sum_{i=1}^n U_i, \quad (2.31)$$

in which  $U_i$  is the strain energy of segment  $i$  of the bar and  $n$  is the number of segments. **Note. This relation holds whether the material behaves in a linear or nonlinear manner.**

Now assume that the material of the bar is linearly elastic and that the internal axial force is constant within each segment. We can then use Eq. (2.30a) to obtain the strain energies of the segments, and Eq. (2.31) becomes

$$U = \sum_{i=1}^n \frac{N_i^2 L_i}{2E_i A_i}, \quad (2.32)$$

in which  $N_i$  is internal axial force acting in segment  $i$  and  $L_i$ ,  $A_i$ , and  $E_i$  are geometrical and mechanical properties of segment  $i$  respectively (product  $EA$  is named as *axial rigidity*).

We can obtain the strain energy of a nonprismatic bar with continuously varying axial force (Fig. 2.24) by applying Eq. (2.30a) to a differential element (shown shaded in the figure) and then integrating along the length of the bar:

$$U = \int_0^L \frac{[N(x)]^2 dx}{2EA(x)}. \quad (2.33)$$

In this equation,  $N(x)$  and  $A(x)$  are the axial force and cross-sectional area at distance  $x$  from the end of the bar.

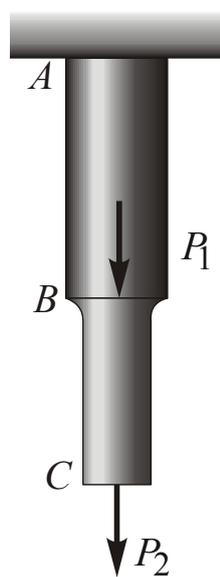


Fig. 2.23 Bar consisting of prismatic segments having different cross-sectional areas and different axial forces

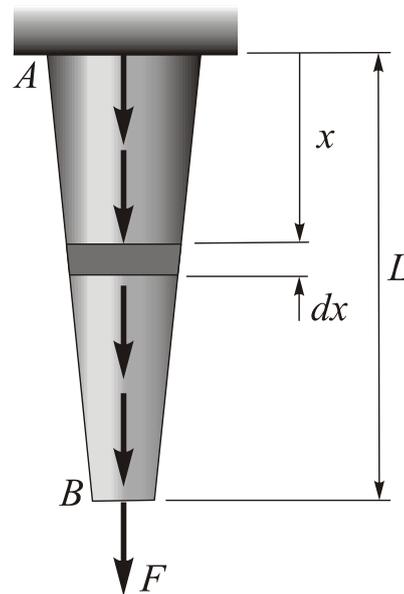


Fig. 2.24 Nonprismatic bar with varying axial force

The expressions (Eqs. 2.30 through 2.33) for strain energy show that *strain energy is not a linear function of the loads*, not even when the material is linearly elastic.

**Note.** It is important to realize that we cannot obtain the strain energy of a structure supporting more than one load by combining the strain energies obtained from the individual loads acting separately.

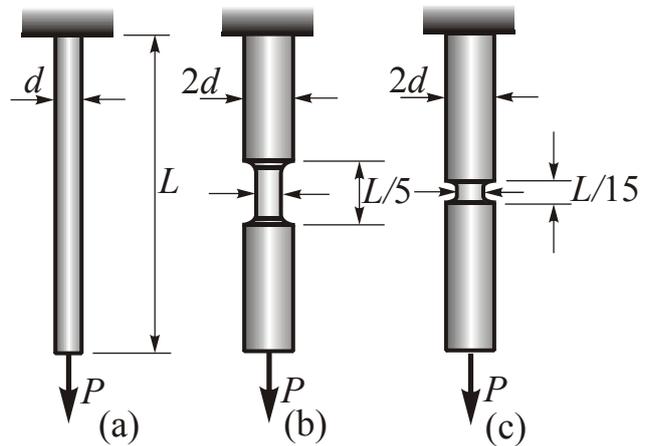
In the case of the nonprismatic bar shown in Fig. 2.23, *the total strain energy is not the sum of the strain energy due to load  $P_1$  acting alone and the strain energy due to load  $P_2$  acting alone*. Instead, we must evaluate the strain energy with all of the loads acting simultaneously.



## EXAMPLES

### Example 2.5

Three round bars having the same length  $L$  but different shapes are shown in the figure. The first bar has diameter  $d$  over its entire length, the second has diameter  $d$  over one-fifth of its length, and the third has diameter  $d$  over one-fifteenth of its length. Elsewhere, the second and third bars have diameter  $2d$ . All three bars are subjected to the same axial load  $P$ .



Compare the amounts of strain energy stored in the bars without consideration the effects of stress concentrations.

**Solution** (1) *Strain energy  $U_1$  of the first bar.* The strain energy of the first bar is found directly from equation:

$$U_1 = \frac{N^2 L}{2EA},$$

in which  $A = \pi d^2 / 4$ ,  $N$  – normal force.

(2) *Strain energy  $U_2$  of the second bar.* The strain energy is found by summing the strain energies in the three segments of the bar:

$$U = \sum_{i=1}^n \frac{N_i^2 L_i}{2E_i A_i}. \quad (a)$$

Normal force  $N_x(x)$  numerically equals to external force  $P$  in accordance with the method of sections.

Thus,

$$U = \sum_{i=1}^n \frac{N_i^2 L_i}{2E_i A_i} = \frac{P^2 (L/5)}{2EA} + \frac{P^2 (4L/5)}{2E(4A)} = \frac{P^2 L}{5EA} = \frac{2U_1}{5}, \quad (b)$$

which is only 40% of the strain energy of the first bar.

**Note.** Increasing the cross-sectional area over part of the length has greatly reduced the amount of strain energy that can be stored in the bar.

(3) Strain energy  $U_3$  of the third bar. Using previous equations, we get

$$U_3 = \sum_{i=1}^n \frac{N_i^2 L_i}{2E_i A_i} = \frac{P^2 (L/15)}{2EA} + \frac{P^2 (14L/15)}{2E(4A)} = \frac{3P^2 L}{20EA} = \frac{3U_1}{10}. \quad (c)$$

The strain energy has now decreased to 30% of the strain energy of the first bar.

**Note.** Comparing these results, we see that the strain energy decreases as the part of the bar with the larger area increases. The same amount of work applied to all three bars will produce the highest stress in the third bar, because the third bar has the least energy-absorbing capacity. If the region having diameter  $d$  is made even smaller, the energy-absorbing capacity will decrease further. We therefore conclude that it takes only a small amount of work to bring the tensile stress to a high value in a bar with a groove, and the narrower the groove, the more severe the condition. When the loads are dynamic and the ability to absorb energy is important, the presence of grooves is very damaging.

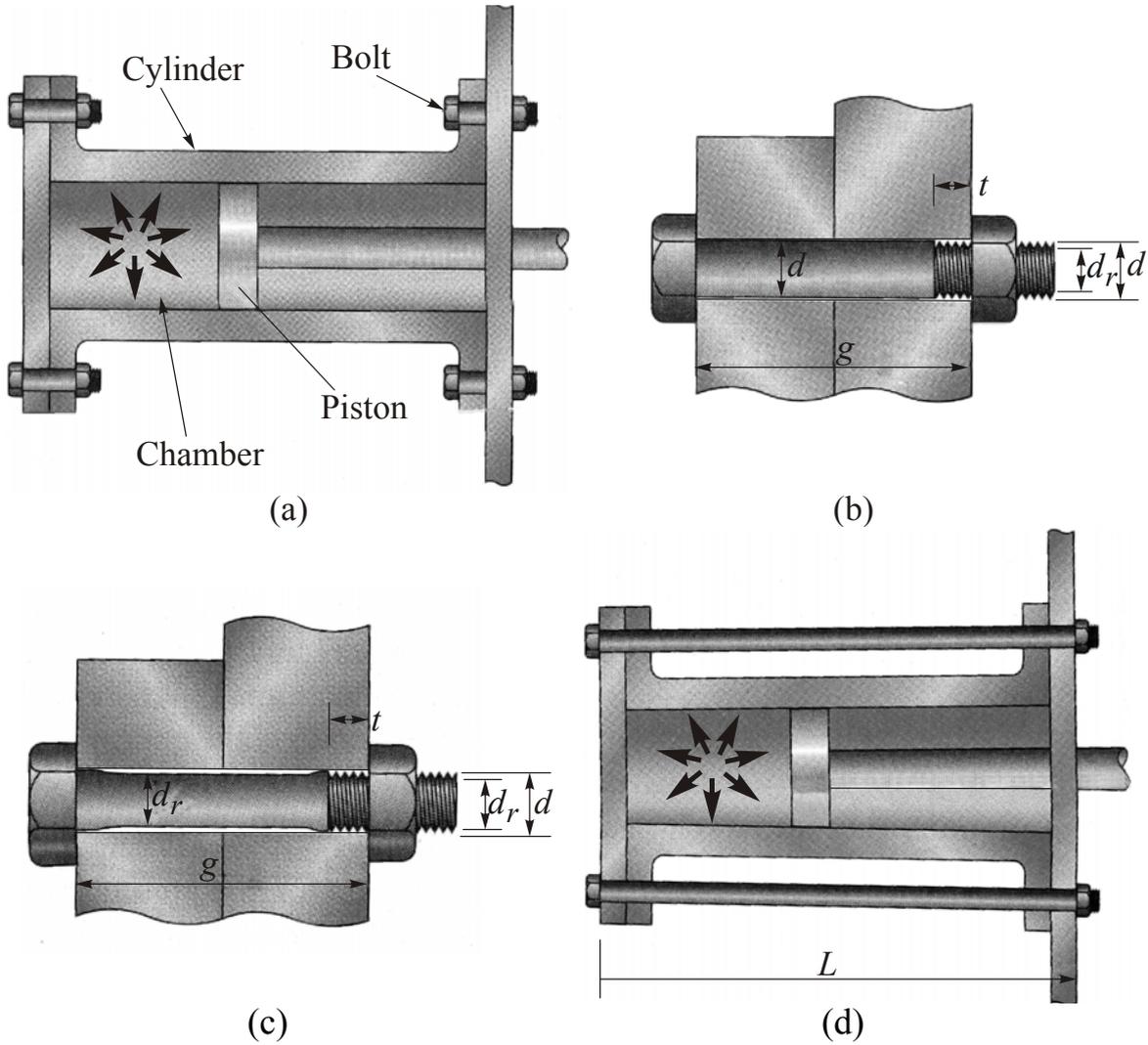
In the case of static loads, the maximum stresses are more important than the ability to absorb energy. In this example, all three bars have the same maximum stress  $N_x / A$ , and therefore all three bars have the same load-carrying capacity when the load is applied statically.

### Example 2.6

The cylinder and cylinder head for a machine are clamped by bolts through the flanges of the cylinder (see figure (a)). A detail of one of the bolts is shown in figure (b). The diameter  $d$  of the shank is 12.7 mm and the root diameter  $d_r$  of the threaded portion is 10.3 mm. The grip  $g$  of the bolts is 38 mm and the threads extend a distance  $t = 6.35$  mm. into the grip. Under the action of repeated cycles of high and low pressure in the chamber, the bolts may eventually break.

To reduce the danger of the bolts failing, the designers suggest two possible modifications: (1) Machine down the shanks of the bolts so that the shank diameter is the same as the thread diameter  $d_r$ , as shown in figure (c). (2) Replace each pair of bolts by a single long bolt, as shown in figure (d). The long bolts are the same as the original bolts (figure (b)) except that the grip is increased to the distance  $L = 343$  mm.

Compare the energy-absorbing capacity of the three bolt configurations: (1) *original bolts*, (2) *bolts with reduced shank diameter*, and (3) *long bolts* (without consideration the effects of stress concentrations). Material is linearly elastic.



(a) Cylinder with piston and clamping bolts; (b) bolt in details; (c) bolt with reduced shank diameter, and (d) bolt with increased length

**Solution** (1) *Original bolts.* The original bolts can be idealized as bars consisting of two segments (Fig. 1b). The left-hand segment has length  $g-t$  and diameter  $d$ , and the right-hand segment has length  $t$  and diameter  $d_r$ . The strain energy of one bolt under a tensile load  $P$  can be obtained by adding the strain energies of the two segments:

$$U_1 = \sum_{i=1}^n \frac{N_i^2 L_i}{2E_i A_i} = \frac{P^2 (g-t)}{2EA_s} + \frac{P^2 t}{2EA_r}, \tag{a}$$

in which  $A_s$  is the cross-sectional area of the shank and  $A_r$  is the cross-sectional area at the root of the threads; thus,

$$A_s = \frac{\pi d^2}{4}, \quad A_r = \frac{\pi d_r^2}{4}. \tag{b}$$

Substituting these expressions into Eq. (a), we get the following formula for the strain energy of one of the original four bolts:

$$U_1 = \frac{2P^2(g-t)}{\pi E d^2} + \frac{2P^2 t}{\pi E d_r^2}. \quad (c)$$

(2) *Bolts with reduced shank diameter.* These bolts can be idealized as prismatic bars having length  $g$  and diameter  $d_r$  (Fig. 2a). Therefore, the strain energy of one bolt is

$$U_2 = \frac{P^2 g}{2EA_r} = \frac{2P^2 g}{\pi E d_r^2}. \quad (d)$$

The ratio of the strain energies for cases (1) and (2) is

$$\frac{U_2}{U_1} = \frac{gd^2}{(g-t)d_r^2 + td^2}, \quad (e)$$

or, upon substituting numerical values,

$$\frac{U_2}{U_1} = \frac{(38.0 \text{ mm})(12.7 \text{ mm})^2}{(38.0 \text{ mm} - 6.35 \text{ mm})(10.3 \text{ mm})^2 + (6.35 \text{ mm})(12.7 \text{ mm})^2} = 1.40.$$

**Note. Using bolts with reduced shank diameters results in a 40% increase in the amount of strain energy that can be absorbed by the bolts. This scheme should reduce the number of failures caused by the impact loads.**

(3) *Long bolts.* The calculations for the long bolts (Fig. 2b) are the same as for the original bolts except the grip  $g$  is changed to the grip  $L$ . Therefore, the strain energy of one long bolt (compare with Eq. (c)) is

$$U_3 = \frac{2P^2(L-t)}{\pi E d^2} + \frac{2P^2 t}{\pi E d_r^2}. \quad (f)$$

Since one long bolt replaces two of the original bolts, we must compare the strain energies by taking the ratio of  $U_3$  to  $2U_1$ , as follows:

$$\frac{U_3}{2U_1} = \frac{(L-t)d_r^2 + td^2}{2(g-t)d_r^2 + 2td^2}. \quad (g)$$

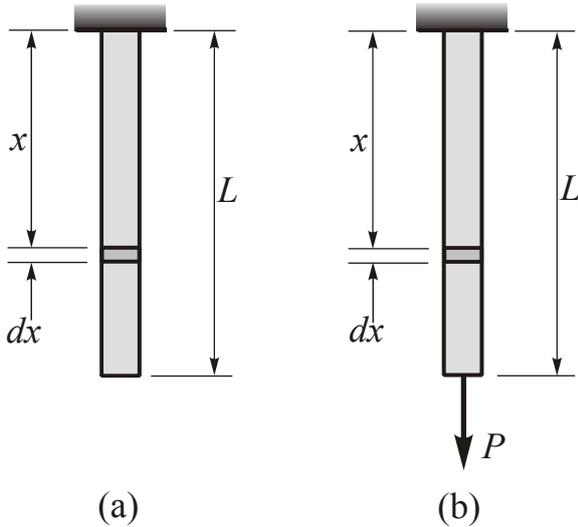
Substituting numerical values gives

$$\frac{U_3}{2U_1} = \frac{(343 \text{ mm} - 6.35 \text{ mm})(10.3 \text{ mm})^2 + (6.35 \text{ mm})(12.7 \text{ mm})^2}{2(38 \text{ mm} - 6.35 \text{ mm})(10.3 \text{ mm})^2 + 2(6.35 \text{ mm})(12.7 \text{ mm})^2} = 4.18.$$

**Note. (1) Using long bolts increases the energy-absorbing capacity by 318% and achieves the greatest safety from the standpoint of strain energy.**

**(2) In perfect designing of the bolts, designers must also consider the maximum tensile stresses, maximum bearing stresses, and stress concentrations.**

**Example 2.7**



(a) Bar loaded by its own weight, and (b) bar loaded by its own weight and also supporting a load  $P$

Determine the strain energy of a prismatic bar suspended from its upper end (see figure). Consider the following loads: (1) the weight of the bar itself, and (2) the weight of the bar plus a load  $P$  at the lower end. Material is linearly elastic.

**Solution** (1) *Strain energy due to the weight of the bar itself* (see figure (a)). The bar is subjected to a varying axial force, the force being zero at the lower end and maximum at the upper end. To determine the axial force, we consider an element of length

$dx$  (shown shaded in the figure) at distance  $x$  from the upper end. The internal axial force  $N(x)$  acting on this element is equal to the weight of the bar below the element:

$$N(x) = \rho g A(L - x), \tag{a}$$

in which  $\rho$  is the density of the material,  $g$  is the acceleration of gravity, and  $A$  is the cross-sectional area of the bar. Substituting into equation

$$U = \int_0^L \frac{[N(x)]^2 dx}{2EA(x)}, \tag{b}$$

and integrating gives the total strain energy

$$U = \int_0^L \frac{[N(x)]^2 dx}{2EA(x)} = \int_0^L \frac{[\rho g A(L - x)]^2 dx}{2EA} = \frac{\rho^2 g^2 AL^3}{6E}. \tag{c}$$

This same result can be obtained from the strain-energy density. At any distance  $x$  from the support, the stress is

$$\sigma = \frac{N(x)}{A} = \rho g(L - x), \tag{d}$$

and strain-energy density is

$$U_0 = \frac{\sigma^2}{2E} = \frac{\rho^2 g^2 (L-x)^2}{2E}. \quad (e)$$

The total strain energy is found by integrating  $U_0$  throughout the volume of the bar:

$$U = \int U_0 dV = \int_0^L U_0 A dx = \int_0^L \frac{\rho^2 g^2 A (L-x)^2}{2E} dx = \frac{\rho^2 g^2 AL^3}{6E}, \quad (f)$$

which agrees with Eq. (c).

(2) *Strain energy due to the weight of the bar plus the load  $P$*  (figure (b)). In this case the axial force  $N(x)$  acting on the element is

$$N(x) = \rho g A (L-x) + P. \quad (g)$$

In result we obtain

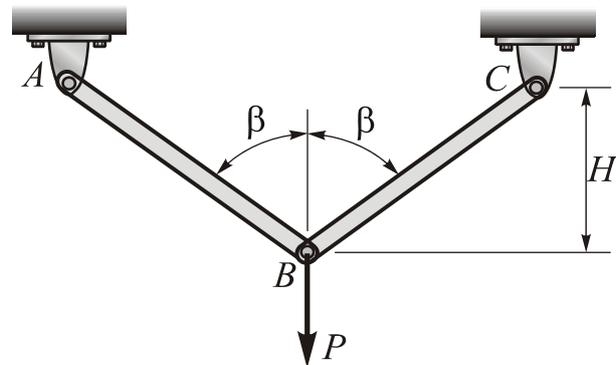
$$U = \int_0^L \frac{[\rho g A (L-x) + P]^2}{2EA} dx = \frac{\rho^2 g^2 AL^3}{6E} + \frac{\rho g PL^2}{2E} + \frac{P^2 L}{2EA}. \quad (h)$$

**Note.** The first term in this expression is the same as the strain energy of a bar hanging under its own weight (Eq. (c)), and the last term is the same as the strain energy of a bar subjected only to an axial force  $P$  (Eq. 2.30a). However, the middle term contains both  $\rho$  and  $P$ , showing that it depends upon both the weight of the bar and the magnitude of the applied load. Thus, this example illustrates that the strain energy of a bar subjected to two loads is not equal to the sum of the strain energies produced by the individual loads acting separately.

### Example 2.8

Determine the vertical displacement  $\delta_B$  of joint  $B$  of the truss shown in the figure. Assume that both members of the truss have the same axial rigidity  $EA$ .

**Solution** Since there is only one load acting on the truss, we can find the displacement corresponding to that load by equating the work of the load to the strain energy of the



Displacement of a truss supporting a single load  $P$

members. However, to find the strain energy we must know the forces in the members (see Eq. 2.30a).

From the equilibrium of forces acting at joint  $B$  we see that the internal axial force  $N(x)$  in either bar is

$$N(x) = \frac{P}{2 \cos \beta}. \quad (\text{a})$$

From the geometry of the truss we know that the length of each bar is  $L_1 = H / \cos \beta$ , where  $H$  is the height of the truss and  $\beta$  is the angle shown in the figure. We can now obtain the strain energy of the two bars:

$$U = 2 \frac{N(x)^2 L_1}{2EA} = \frac{P^2 H}{4EA \cos^3 \beta}. \quad (\text{b})$$

The work of the load  $P$  is

$$W = \frac{P \delta_B}{2}, \quad (\text{c})$$

where  $\delta_B$  is the linear displacement of joint  $B$ . Equating  $U$  and  $W$  and solving for  $\delta_B$ , we obtain

$$\delta_B = \frac{PH}{2EA \cos^3 \beta}. \quad (\text{d})$$

This equation gives the vertical displacement of joint  $B$  of the truss.

**Note.** We found the displacement using only equilibrium and strain energy.



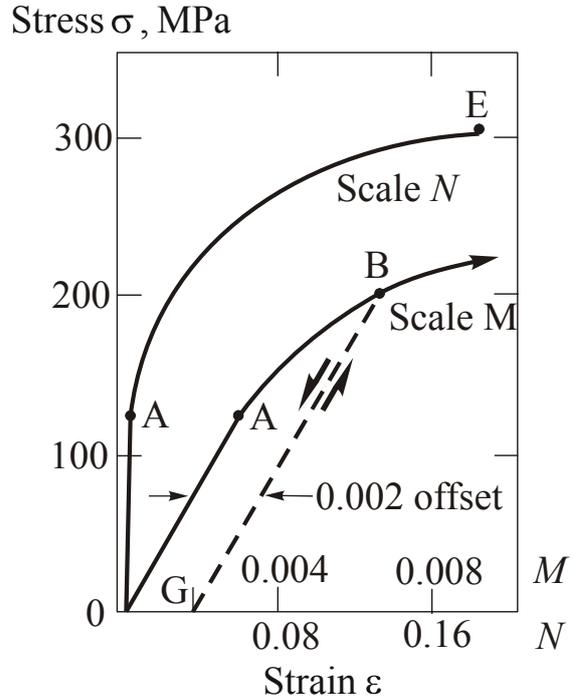
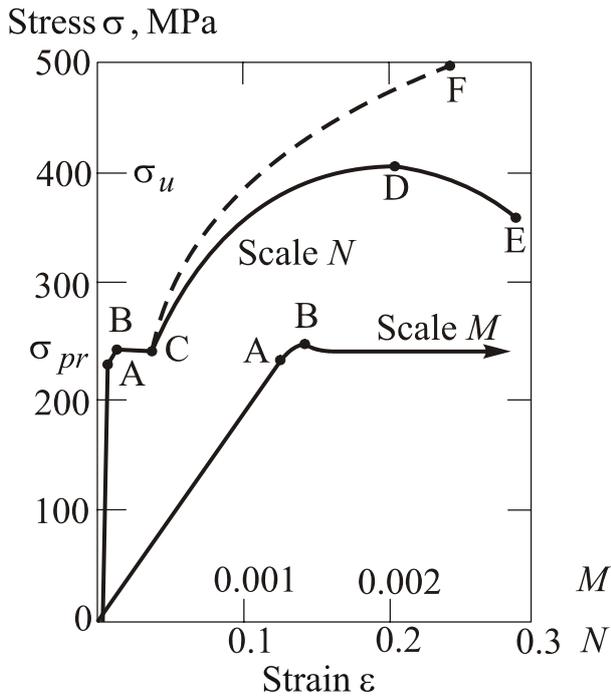
## PROBLEMS

**Problem 2.66** Calculate the modulus of resilience for two grades of steel: (1) ASTM-A242 (yield strength in tension  $\sigma_y = 345$  MPa, modulus of elasticity  $E = 200$  GPa) and (2) cold-rolled, stainless steel (302) (yield strength in tension  $\sigma_y = 517$  MPa, modulus of elasticity  $E = 193$  GPa).

**Problem 2.67** Calculate the modulus of resilience for the following two materials: (1) aluminum alloy 2014-T6 (yield strength in tension  $\sigma_y = 413.7$  MPa, modulus of elasticity  $E = 73$  GPa) and (2) annealed yellow brass (yield strength in tension  $\sigma_y = 103.4$  MPa, modulus of elasticity  $E = 103$  GPa).

**Problem 2.68** Using the stress-strain diagram of a structural steel shown in the figure, determine: (1) the modulus of resilience and (2) the approximate modulus of toughness.

**Problem 2.69** From the stress-strain curve of a magnesium alloy shown in the figure, determine: (1) the modulus of resilience and (2) the approximate modulus of toughness.



### Chapter 3 Two-Dimensional (Plane) Stress State

A two-dimensional state of stress exists at a point of deformable solid, when the stresses are independent of one of the two coordinate axis. It means that *the general feature of this type of stress state is the presence of one zero principal plane*. Examples include the stresses arising on inclined sections of an axially loaded rod (Fig. 3.1), a shaft in torsion (Fig. 3.2), a beam with transversely applied force (Fig. 3.3), a beam at combined loading (Fig. 3.4), and also thin-walled vessel under internal pressure  $p$  (Fig. 3.5).

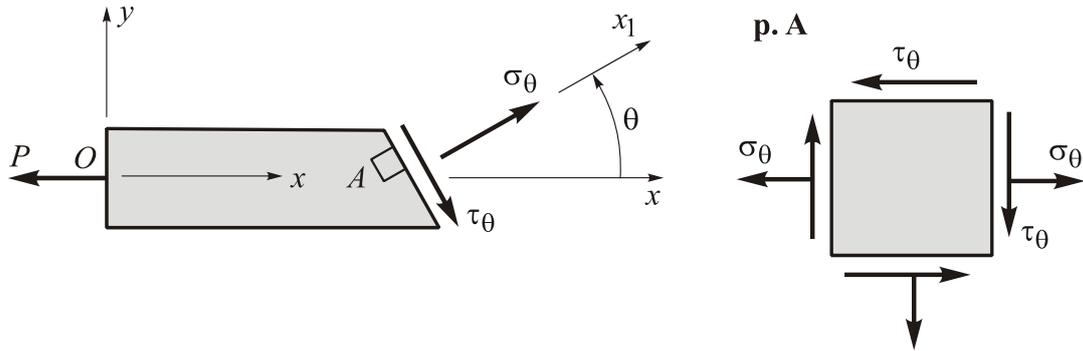


Fig. 3.1 Two-dimensional stresses on inclined section in axial loading

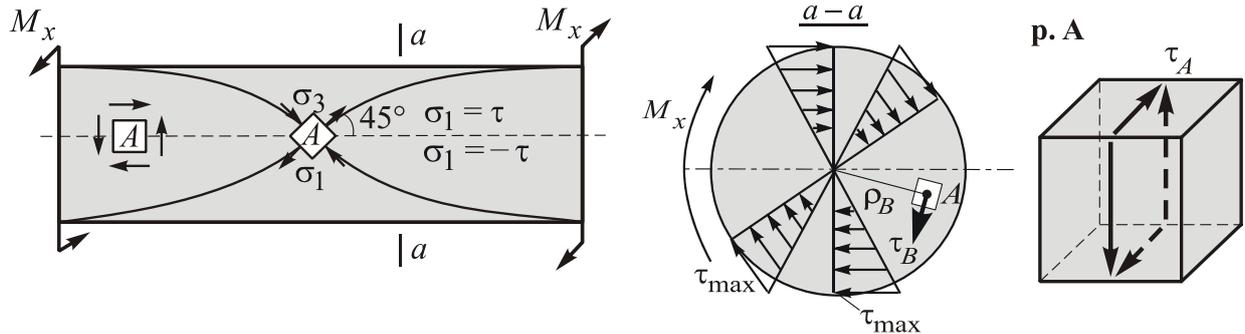


Fig. 3.2 Two-dimensional stress state at an arbitrary point A of the shaft surface and at an arbitrary point B in  $a-a$  cross section in torsion. The stresses are given by the torsional formula  $\tau(\rho) = M_x \rho / I \rho$

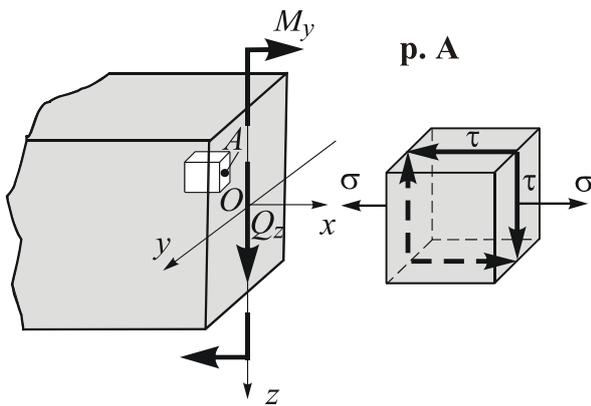


Fig. 3.3 Two-dimensional stress state at a point A of the rod in transverse bending. The stresses are given by the bending and shear formulas:

$$\sigma(z) = Mz / I_y,$$

$$\tau(z) = Q_z S_y^* / (bI_y)$$

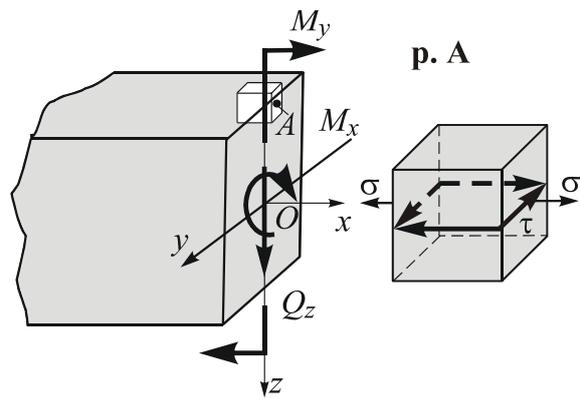


Fig. 3.4 Two-dimensional stress state at a point A of the rod in combined loading

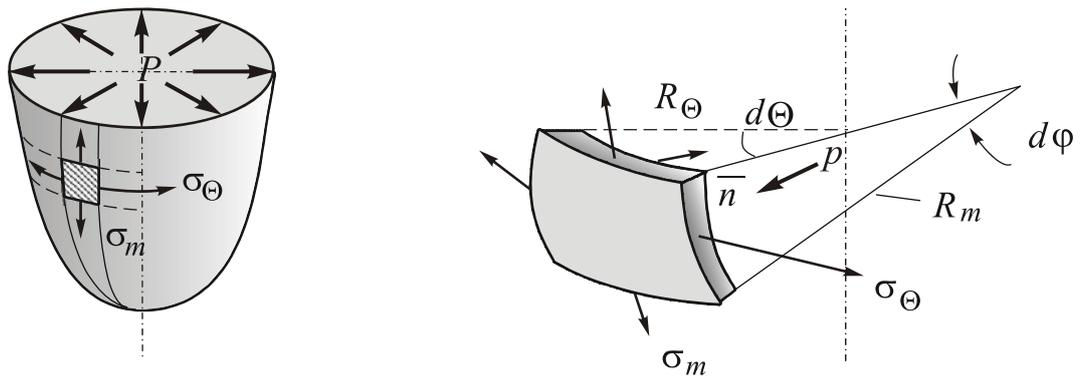


Fig. 3.5 Element in biaxial stress state in pressure vessel

Two-dimensional problems are of two classes: *plane stress* and *plane strain*. The condition that occurs in a *thin plate* subjected to loading uniformly distributed over the thickness and parallel to the plane of plate typifies the *state of plane stress* (plane stressed state, plane stress) (Fig. 3.6). Because the plate is thin, the stress-distribution may be closely approximated by assuming that two-dimensional stress components do not vary throughout the thickness and the other components are zero. Another case of plane stress exists on the free surface of a structural or machine component.

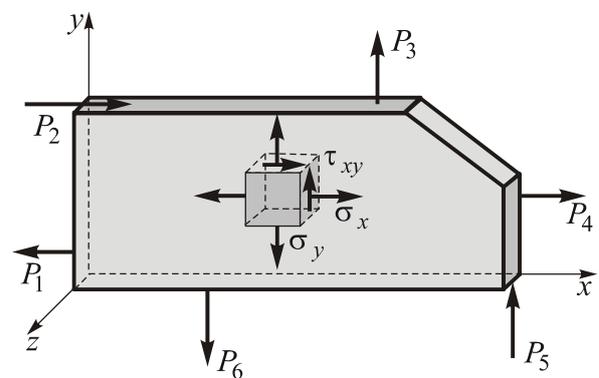


Fig. 3.6 Thin plane subjected to plane stress

To explain plane stress, we will consider the stress element shown in Fig. 3.7. This element is infinitesimal in size and can be sketched either as a cube or as a rectangular parallelepiped.

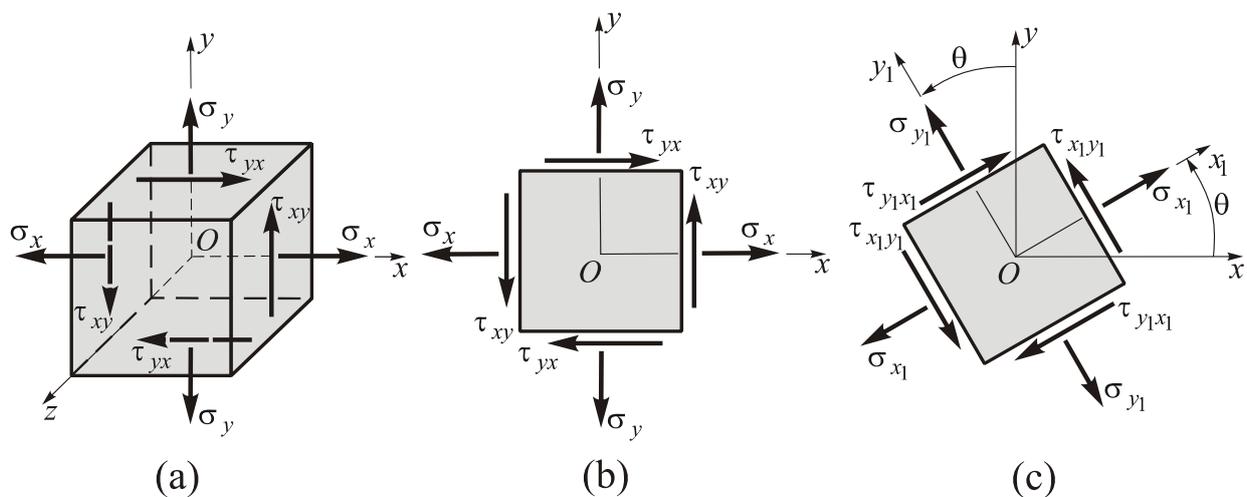


Fig. 3.7 Elements in plane stress: (a) three-dimensional view of an element oriented to the  $xyz$  axes, (b) two-dimensional view of the same element, and (c) two-dimensional view of an element oriented to the  $x_1y_1z$  axes

The  $xyz$  axes are parallel to the edges of the element, and the faces of the element are designated by the directions of their outward normals. For instance, the *right-hand face* of the element is referred to as the *positive  $x$  face*, and the left-hand face (hidden from the viewer) is referred to as the *negative  $x$  face*. Similarly, the *top face is the positive  $y$  face*, and the front face is the *positive  $z$  face*.

When the material is in plane stress in the  $xy$  plane, only the  $x$  and  $y$  faces of the element are subjected to stresses, and all stresses act parallel to the  $x$  and  $y$  axes, as shown in Fig. 3.7a. This stress condition is very common because *it exists at the surface of any stressed body*, except at points where external loads act on the surface. When the element shown in Fig. 3.7a is located at the free surface of a body, the  $z$  face is in the plane of the surface (no stresses) and the  $z$  axis is normal to the surface. This face may be considered as *zero principal plane*.

The symbols for the stresses shown in Fig. 3.7a have the following meanings. A normal stress  $\sigma$  has a subscript that identifies the face on which the stress acts; for instance, the stress  $\sigma_x$  acts on the  $x$  face of the element and the stress  $\sigma_y$  acts on the  $y$  face of the element. *Since the element is infinitesimal in size, equal normal stresses act on the opposite faces*. The sign convention for normal stresses is the familiar one, namely, tension is positive and compression is negative.

A shear stress  $\tau$  has two subscripts – the first subscript denotes the normal to the face on which the stress acts, and the second gives the direction on that face. Thus, the stress  $\tau_{xy}$  acts on the  $x$  face in the direction of the  $y$  axis (Fig. 3.7a), and stress  $\tau_{yx}$  acts on the  $y$  face in the direction of the  $x$  axis.

The sign convention for shear stresses is as follows. *A shear stress is positive when it acts on a positive face of an element in the positive direction of an axis, and it is negative when it acts on a positive face of an element in the negative direction of an axis*. Therefore, the stresses  $\tau_{xy}$  and  $\tau_{yx}$  shown on the positive  $x$  and  $y$  faces in Fig. 3.7a are positive shear stresses. Similarly, on a negative face of the element, a shear stress is positive when it acts in the negative direction of an axis. Hence, the stresses  $\tau_{xy}$  and  $\tau_{yx}$  shown on the negative  $x$  and  $y$  faces of the element are also positive.

The preceding sign convention for shear stresses is dependable on the equilibrium of the element, because we know that shear stresses on opposite faces of an infinitesimal element must be equal in magnitude and opposite in direction. Hence, according to our sign convention, a positive stress  $\tau_{xy}$  acts upward on the positive face (Fig. 3.7a) and downward on the negative face. In a similar manner, the stresses  $\tau_{yx}$  acting on the top and bottom faces of the element are positive although they have opposite directions.

We know that shear stresses on mutually perpendicular planes are equal in magnitude and have directions such that *both stresses point toward, or both point away from, the line of intersection of the faces*. Inasmuch as  $\tau_{xy}$  and  $\tau_{yx}$  are positive in the directions shown in the Fig. 3.7, they are consistent with this observation. Therefore, we note that

$$\tau_{xy} = \tau_{yx}. \quad (3.1)$$

This equation is called the *law of equality of shear stresses*. It was derived from equilibrium of the element.

For convenience in sketching plane-stress elements, we usually draw only a two-dimensional view of the element, as shown in Fig. 3.7b.

### 3.1 Stresses on Inclined Planes

Our goal now is to consider the stresses acting on inclined sections, assuming that the stresses  $\sigma_x$ ,  $\sigma_y$ , and  $\tau_{xy}$  (Figs. 3.7a and b) are known. To determine the stresses acting on an inclined section at positive (counterclockwise)  $\theta$ -angle, we consider a new stress element (Fig. 3.7c) that is located at the same point in the material as the original element (Fig. 3.7b). However, the new element has faces that are parallel and perpendicular to the inclined direction. Associated with this new element are axes  $x_1$ ,  $y_1$  and  $z_1$  such that the  $z_1$  axis coincides with the  $z$  axis and the  $x_1, y_1$  axes are rotated counterclockwise through an angle  $\theta$  with respect to the  $xy$  axes. The normal and shear stresses acting on this new element are denoted  $\sigma_{x_1}$ ,  $\sigma_{y_1}$ ,  $\tau_{x_1y_1}$ , and  $\tau_{y_1x_1}$ , using the same subscript designations and sign conventions described above for the stresses acting on the  $xy$  element. The previous conclusions regarding the shear stresses still apply, so that

$$\tau_{x_1y_1} = \tau_{y_1x_1}. \quad (3.2)$$

From this equation and the equilibrium of the element, we see that the *shear stresses acting on all four side faces of an element in plane stress are known if we determine the shear stress acting on any one of those faces*.

The stresses acting on the inclined  $x_1, y_1$  element (Fig. 3.7c) can be expressed in terms of the stresses on the  $xy$  element (Fig. 3.7b) by using *equations of equilibrium*. For this purpose, we choose a *wedge-shaped stress element* (Fig. 3.8a) having an inclined face that is the same as the  $x_1$  face of the inclined element. The other two side faces of the wedge are parallel to the  $x$  and  $y$  axes.

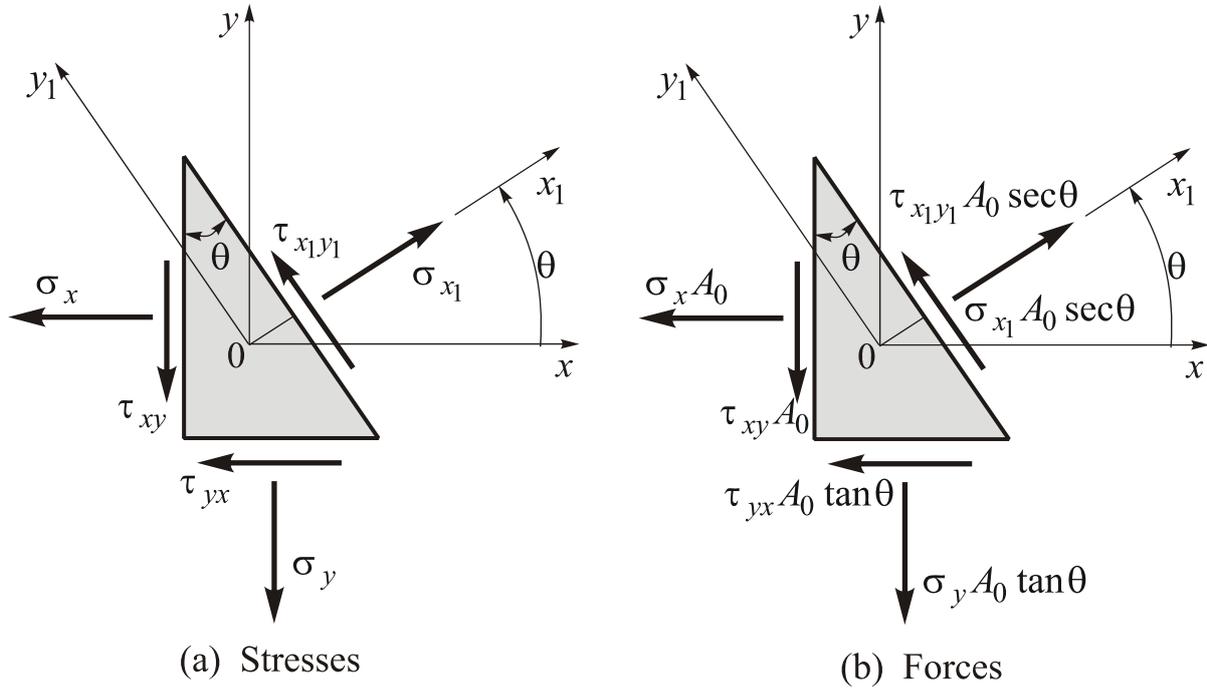


Fig. 3.8 Wedge-shaped stress element in plane stress state: (a) stresses acting on the element, and (b) internal forces acting on the element

In order to write equations of equilibrium for the wedge, we need to construct a free-body diagram showing the forces acting on the faces. Let us denote the area of the left-hand side face (that is, the negative  $x$  face) as  $A_0$ . Then the normal and shear forces acting on that face are  $\sigma_x A_0$  and  $\tau_{xy} A_0$ , as shown in the free-body diagram of Fig. 3.8b. The area of the bottom face (or negative face) is  $A_0 \tan \theta$ , and the area of the inclined face (or positive  $x_1$  face) is  $A_0 \sec \theta$ . Thus, the normal and shear forces acting on these faces have the magnitudes and directions shown in the Fig. 3.8 b.

The forces acting on the left-hand and bottom faces can be resolved into orthogonal components acting in the  $x_1$  and  $y_1$  directions. Then we can obtain two equations of equilibrium by summing forces in those directions. The first equation, obtained by summing forces in the  $x_1$  direction, is

$$\begin{aligned} \sigma_{x_1} A_0 \sec \theta - \sigma_x A_0 \cos \theta - \tau_{xy} A_0 \sin \theta - \\ - \sigma_y A_0 \tan \theta \sin \theta - \tau_{yx} A_0 \tan \theta \cos \theta = 0. \end{aligned} \quad (3.3)$$

Summation of forces in the  $y_1$  direction gives

$$\begin{aligned} \tau_{x_1 y_1} A_0 \sec \theta + \sigma_x A_0 \sin \theta - \tau_{xy} A_0 \cos \theta - \\ - \sigma_y A_0 \tan \theta \cos \theta + \tau_{yx} A_0 \tan \theta \sin \theta = 0. \end{aligned} \quad (3.4)$$

Using the relationship  $\tau_{xy} = \tau_{yx}$ , we obtain after simplification the following two equations:

$$\sigma_{x_1} = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + 2\tau_{xy} \sin \theta \cos \theta, \quad (3.5)$$

$$\tau_{x_1 y_1} = -(\sigma_x - \sigma_y) \sin \theta \cos \theta + \tau_{xy} (\cos^2 \theta - \sin^2 \theta). \quad (3.6)$$

Equations (3.5) and (3.6) give the normal and shear stresses acting on the  $x_1$  plane in terms of the angle  $\theta$  and the stresses  $\sigma_x$ ,  $\sigma_y$ , and  $\tau_{xy}$  acting on the  $x$  and  $y$  planes.

It is interesting to note, that in  $\theta = 0$  Eqs. (3.5) and (3.6) give  $\sigma_{x_1} = \sigma_x$  and  $\tau_{x_1 y_1} = \tau_{xy}$ . Also, when  $\theta = 90^\circ$ , these equations give  $\sigma_{x_1} = \sigma_y$  and  $\tau_{x_1 y_1} = -\tau_{xy} = -\tau_{yx}$ . In the latter case, since the  $x_1$  axis is vertical when  $\theta = 90^\circ$ , the stress  $\tau_{x_1 y_1}$  will be positive when it acts to the left. However, the stress  $\tau_{yx}$  acts to the right, and therefore  $\tau_{x_1 y_1} = -\tau_{yx}$ .

Equations (3.5) and (3.6) can be expressed in a more convenient form by introducing the following trigonometric identities:

$$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta), \quad \sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta), \quad \sin \theta \cos \theta = \frac{1}{2} \sin 2\theta. \quad (3.7)$$

After these substitutions the equations become

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta, \quad (3.8)$$

$$\tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta. \quad (3.9)$$

These equations are known as the *transformation equations for plane stress* because they transform the stress components from one set of axes to another.

**Note. (1) Only one intrinsic state of stress exists at the point in a stressed body, regardless of the orientation of the element, i.e. whether represented by stresses acting on the  $xy$  element (Fig. 3.7b) or by stresses acting on the inclined  $x_1 y_1$  element (Fig. 3.7c). (2) Since the transformation equations were derived only from equilibrium of an element, they are applicable to stresses in any kind of material, whether linear or nonlinear, elastic or inelastic.**

An important result concerning the normal stresses can be obtained from the transformation equations. The normal stress  $\sigma_{y_1}$  acting on the  $y_1$  face of the

inclined element (Fig. 3.7c) can be obtained from Eq. (3.8) by substituting  $\theta + 90^\circ$  for  $\theta$ . The result is the following equation for  $\sigma_{y_1}$ :

$$\sigma_{y_1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta. \quad (3.10)$$

Summing the expressions for  $\sigma_{x_1}$  and  $\sigma_{y_1}$  (Eqs. 3.8 and 3.10), we obtain the following equation for plane stress:

$$\sigma_{x_1} + \sigma_{y_1} = \sigma_x + \sigma_y = \text{const.} \quad (3.11)$$

**Note. The sum of the normal stresses acting on perpendicular faces of plane-stress elements (at a given point in a stressed body) is constant and independent of the angle  $\theta$ .**

The graphs of the normal and shear stresses varying are shown in Fig. 3.9, which are the graphs of  $\sigma_{x_1}$  and  $\tau_{x_1y_1}$  versus the angle  $\theta$  (from Eqs. 3.8 and 3.9). The graphs are plotted for the particular case of  $\sigma_y = 0.2\sigma_x$  and  $\tau_{xy} = 0.8\sigma_x$ . It is seen from the plots that the stresses vary continuously as the orientation of the element is changed. At certain angles, the normal stress reaches a maximum or minimum value; at other angles, it becomes zero. Similarly, the shear stress has maximum, minimum, and zero values at certain angles.

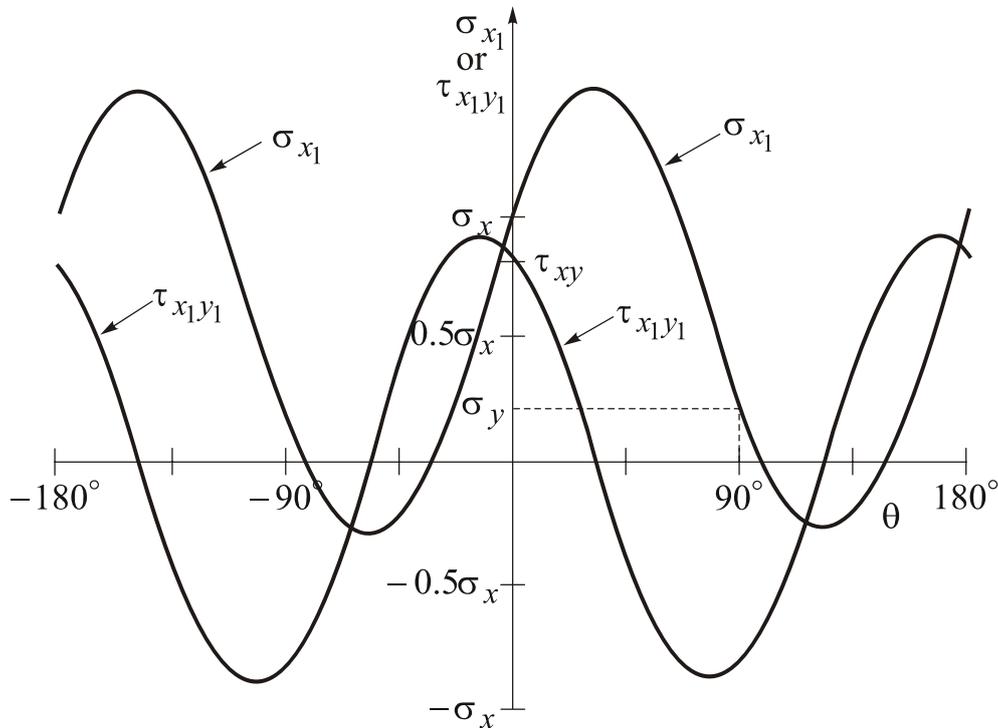


Fig. 3.9 Graphs of normal stress  $\sigma_{x_1}$  and shear stress  $\tau_{x_1y_1}$  versus the angle  $\theta$  (for  $\sigma_y = 0.2\sigma_x$  and  $\tau_{xy} = 0.8\sigma_x$ )

## Special Cases of Plane Stress

### 3.2.1 Uniaxial Stress State as a Simplified Case of Plane Stress

The general case of plane stress reduces to simpler states of stress under special conditions. For instance, as previously discussed, if all stresses acting on the  $xy$  element (Fig. 3.7b) are zero except for the normal stress  $\sigma_x$ , then the element is in *uniaxial stress* (Fig. 3.10). The corresponding transformation equations, obtained by setting  $\sigma_y$  and  $\tau_{xy}$  equal to zero in Eqs. (3.8) and (3.9), are

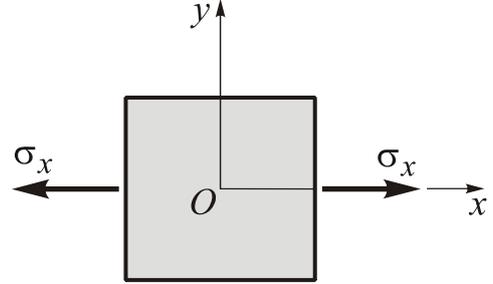


Fig. 3.10 Element in uniaxial stress

$$\sigma_{x_1} = \frac{\sigma_x}{2}(1 + \cos 2\theta) = \sigma_x \cos^2 \theta, \quad (3.12)$$

$$\tau_{x_1y_1} = -\frac{\sigma_x}{2}(\sin 2\theta). \quad (3.13)$$

### 3.2.2 Pure Shear as a Special Case of Plane Stress

*Pure shear* is another special case of plane stress state (Fig. 3.11), for which the transformation equations are obtained by substituting  $\sigma_x = 0$  and  $\sigma_y = 0$  into Eqs. (3.8) and (3.9):

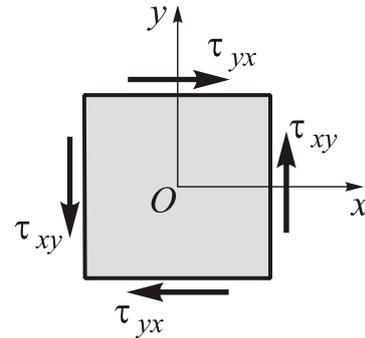


Fig. 3.11 Element in pure shear

$$\sigma_{x_1} = \tau_{xy} \sin 2\theta, \quad (3.14)$$

$$\tau_{x_1y_1} = \tau_{xy} \cos 2\theta. \quad (3.15)$$

### 3.2.3 Biaxial Stress

The next special case of plane stress state is called *biaxial stress*, in which the  $xy$  element is subjected to normal stresses in both the  $x$  and  $y$  directions but *without any shear stresses* (Fig. 3.12). The equations for biaxial stress are obtained from Eqs. (3.8) and (3.9) simply by dropping the terms containing  $\tau_{xy}$ :

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta, \quad (3.16)$$

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta. \quad (3.17)$$

Biaxial stress occurs in many kinds of structures, including *thin-walled pressure vessels* (see Fig. 3.13).

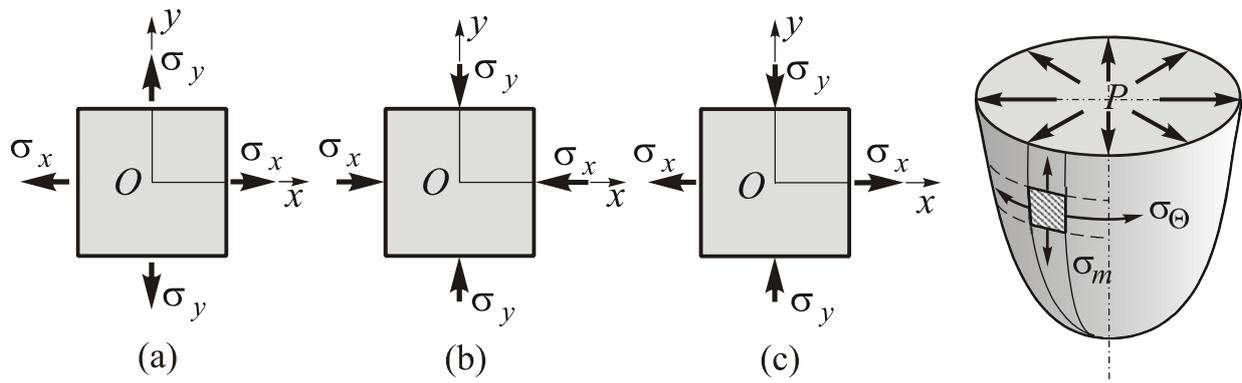


Fig. 3.12 Elements in biaxial stress

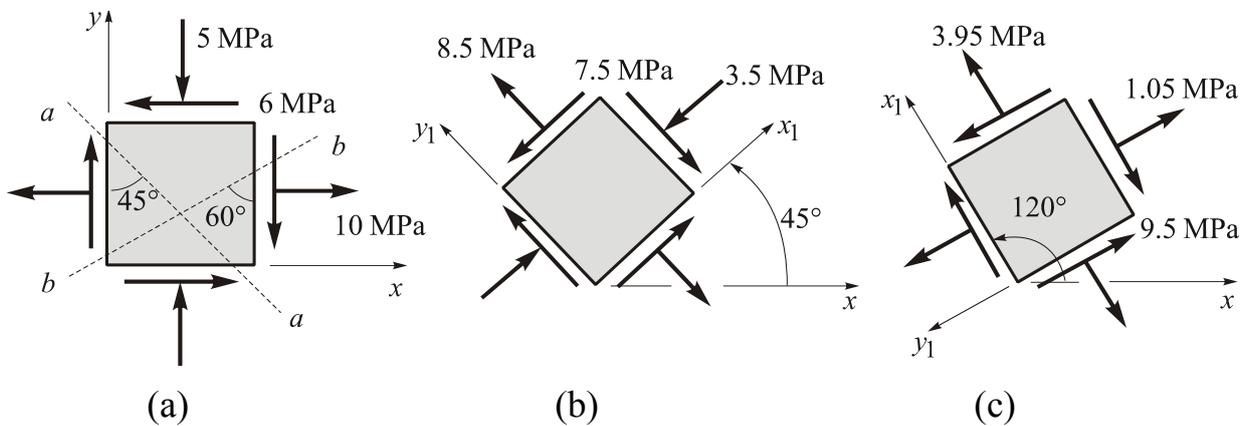
Fig. 3.13 Element in biaxial stress state in pressure vessel (stresses, normal to the surface are assumed to be zero)



### EXAMPLES

#### Example 3.1

The state of stress at a point in the machine element is shown in Fig. a. Determine the normal and shearing stresses acting on an inclined plane parallel to (1) line  $a - a$  and (2) line  $b - b$ .



**Solution** The  $x_1$  direction is that of a normal to the inclined plane. We want to obtain the transformation of stress from the  $xy$  system of coordinates to the  $x_1y_1$  system.

**Note, that the stresses and the rotations must be designated with their correct signs.**

(1) Applying Eqs. (3.8 through 3.10) for  $\theta = 45^\circ$ ,  $\sigma_x = 10$  MPa,  $\sigma_y = -5$  MPa, and  $\tau_{xy} = -6$  MPa, we obtain

$$\sigma_{x_1} = \frac{1}{2}(10 - 5) + \frac{1}{2}(10 + 5)\cos 90^\circ - 6\sin 90^\circ = -3.5 \text{ MPa},$$

$$\tau_{x_1y_1} = -\frac{1}{2}(10 + 5)\sin 90^\circ - 6\cos 90^\circ = -7.5 \text{ MPa},$$

and

$$\sigma_{y_1} = \frac{1}{2}(10 - 5) - \frac{1}{2}(10 + 5)\cos 90^\circ + 6\sin 90^\circ = 8.5 \text{ MPa}.$$

The results are indicated in Fig. b.

(2) As  $\theta = 30 + 90 = 120^\circ$ , from Eqs. (3.8 through 3.10), we have

$$\sigma_{x_1} = \frac{1}{2}(10 - 5) + \frac{1}{2}(10 + 5)\cos 240^\circ - 6\sin 240^\circ = 3.95 \text{ MPa},$$

$$\tau_{x_1y_1} = -\frac{1}{2}(10 + 5)\sin 240^\circ - 6\cos 240^\circ = 9.5 \text{ MPa},$$

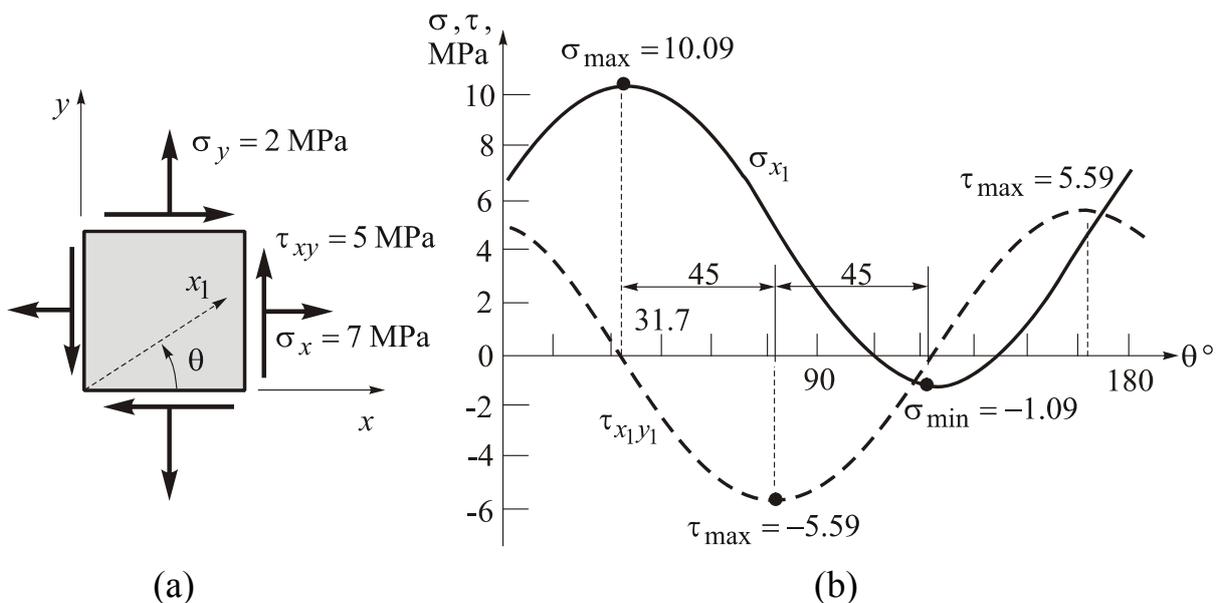
and

$$\sigma_{y_1} = \frac{1}{2}(10 - 5) - \frac{1}{2}(10 + 5)\cos 240^\circ + 6\sin 240^\circ = 1.05 \text{ MPa}.$$

The results are indicated in Fig. c.

### Example 3.2

A two-dimensional stress state at a point in a loaded structure is shown in Fig. a. (1) Write the stress-transformation equations. (2) Compute  $\sigma_{x_1}$  and  $\tau_{x_1y_1}$  with  $\theta$  between  $0$  and  $180^\circ$  in  $15^\circ$  increments for  $\sigma_x = 7$  MPa,  $\sigma_y = 2$  MPa, and  $\tau_{xy} = 5$  MPa. Plot the graphs  $\sigma_{x_1}(\theta)$  and  $\tau_{x_1y_1}(\theta)$ .



Variation in normal stress  $\sigma_{x_1}$  and shearing stress  $\tau_{x_1y_1}$  with angle  $\theta$  varying between  $0$  and  $180^\circ$

**Solution** (1) We express Eqs. (3.8) and (3.9) as follows:

$$\sigma_{x_1} = A + B \cos 2\theta + C \sin 2\theta,$$

$$\tau_{x_1y_1} = -B \sin 2\theta + C \cos 2\theta,$$

where

$$A = \frac{1}{2}(\sigma_x + \sigma_y), \quad B = \frac{1}{2}(\sigma_x - \sigma_y), \quad C = \tau_{xy}.$$

(2) Substitution of the prescribed values into Eqs. (3.8) and (3.9) results in

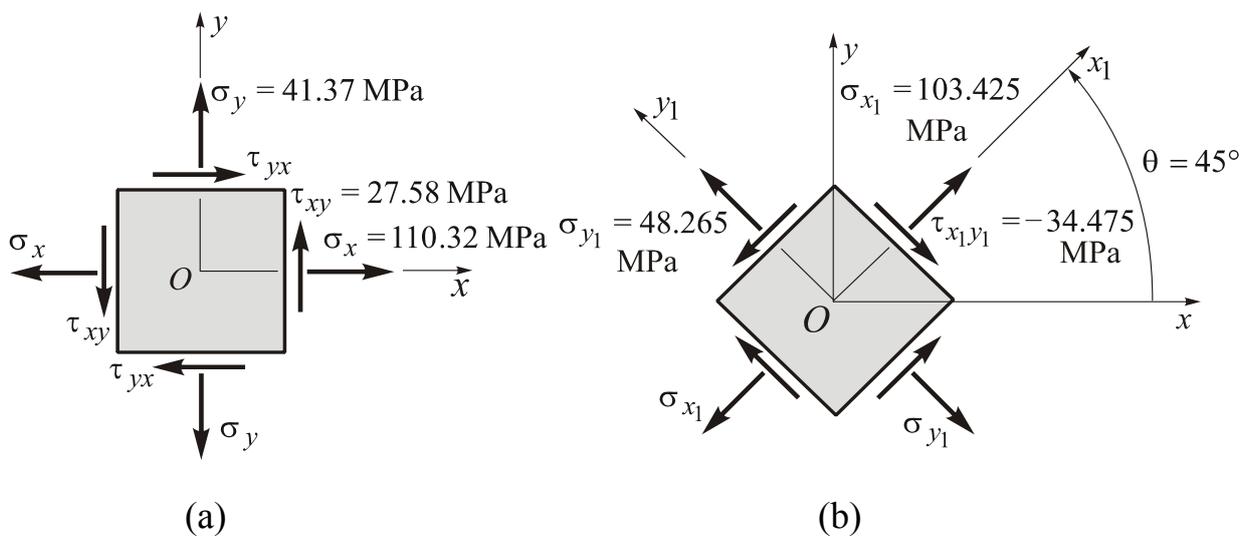
$$\sigma_{x_1} = 4.5 + 2.5 \cos 2\theta + 5 \sin 2\theta,$$

$$\tau_{x_1y_1} = -2.5 \sin 2\theta + 5 \cos 2\theta.$$

Here, permitting  $\theta$  to vary from  $0$  to  $180^\circ$  in increments of  $15^\circ$  yields the data upon which the curves shown in Fig. b are based. These cartesian representations indicate how the stresses vary around a point. Observe that the *direction of maximum (and minimum) shear stress bisects the angle between the maximum and minimum normal stresses*. Moreover, the normal stress is either a maximum or a minimum on planes  $\theta = 31.7^\circ$  and  $\theta = 31.7^\circ + 90^\circ$ , respectively, for which the shearing stress is zero. **Note. The conclusions drawn from the foregoing are valid for any state of stress.**

### Example 3.3

An element in plane stress is subjected to stresses  $\sigma_x = 110.32$  MPa,  $\sigma_y = 41.37$  MPa, and  $\tau_{xy} = \tau_{yx} = 27.58$  MPa, as shown in Fig. a. Determine the stresses acting on an element inclined at an angle  $\theta = 45^\circ$ .



(a) Element in plane stress, and (b) element inclined at an angle  $\theta = 45^\circ$

**Solution** To determine the stresses acting on an inclined element, we will use the transformation equations (Eqs. 3.8 and 3.9). From the given numerical data, we obtain the following values for substitution into those equations:

$$\frac{\sigma_x + \sigma_y}{2} = 75.845 \text{ MPa}, \quad \frac{\sigma_x - \sigma_y}{2} = 34.475 \text{ MPa}, \quad \tau_{xy} = 27.58 \text{ MPa},$$

$$\sin 2\theta = \sin 90^\circ = 1, \quad \cos 2\theta = \cos 90^\circ = 0.$$

Substituting these values into Eqs. 3.8 and 3.9, we get

$$\begin{aligned} \sigma_{x_1} &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta = \\ &= 75.845 \text{ MPa} + (34.475 \text{ MPa})(0) + (27.58 \text{ MPa})(1) = 103.425 \text{ MPa}, \\ \tau_{x_1 y_1} &= -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta = \\ &= -(34.475 \text{ MPa})(1) + (27.58 \text{ MPa})(0) = -34.475 \text{ MPa}. \end{aligned}$$

In addition, the stress  $\sigma_{y_1}$  may be obtained from Eq. (3.10):

$$\begin{aligned} \sigma_{y_1} &= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta = \\ &= 75.845 \text{ MPa} - (34.475 \text{ MPa})(0) - (27.58 \text{ MPa})(1) = 48.265 \text{ MPa}. \end{aligned}$$

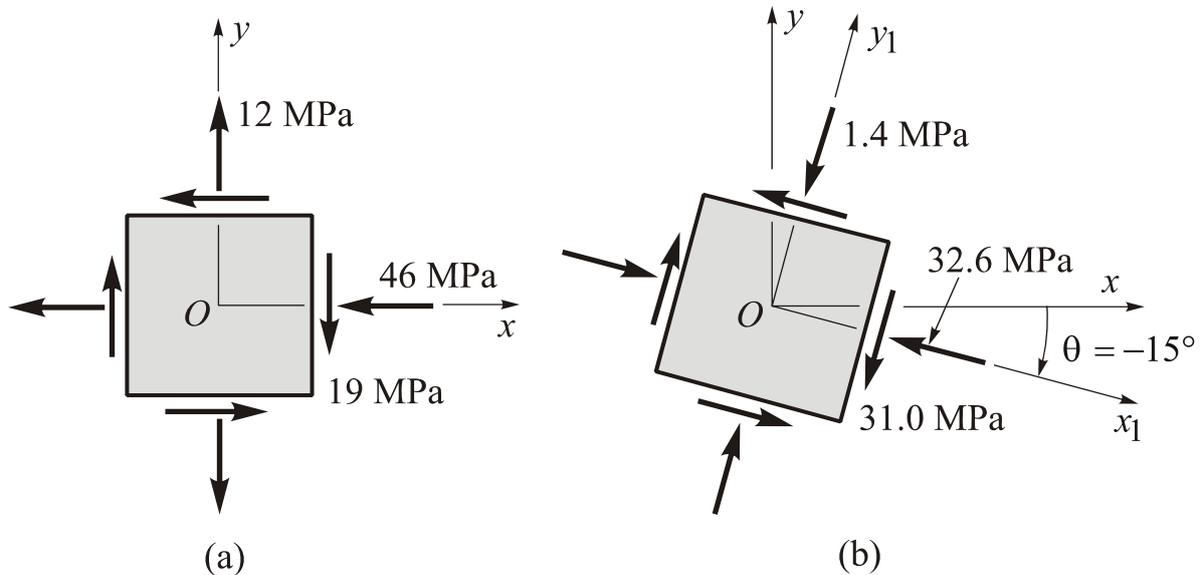
From these results we can obtain the stresses acting on all sides of an element oriented at  $\theta = 45^\circ$ , as shown in Fig. b. The arrows show the true directions in which the stresses act. Note especially the directions of the shear stresses, all of which have the same magnitude. Also, observe that the sum of the normal stresses remains constant and equal to 151.69 MPa from Eq. (3.11):

$$\sigma_{x_1} + \sigma_{y_1} = \sigma_x + \sigma_y = 151.69 \text{ MPa}.$$

**Note.** The stresses shown in Fig. b represent the same intrinsic state of stress as do the stresses shown in Fig. a. However, the stresses have different values because the elements on which they act have different orientations.

### Example 3.4

On the surface of a loaded structure a plane stress state exists at a point, where the stresses have the magnitudes and directions shown on the stress element of Fig. a. Determine the stresses acting on an element that is oriented at a clockwise angle of  $15^\circ$  with respect to the original element.



**Solution** The stresses acting on the original element (see Fig. a) have the following values:

$$\sigma_x = -46 \text{ MPa}, \quad \sigma_y = 12 \text{ MPa}, \quad \tau_{xy} = -19 \text{ MPa}.$$

An element oriented at a clockwise angle of  $15^\circ$  is shown in Fig. b, where the  $x_1$  axis is at an angle  $\theta = -15^\circ$  with respect to the  $x$  axis (clockwise rotation).

We will calculate the stresses on the  $x_1$  face of the element oriented at  $\theta = -15^\circ$  by using the transformation equations (Eqs. (3.8) and (3.9)). The components are:

$$A = \frac{\sigma_x + \sigma_y}{2} = -17 \text{ MPa}, \quad B = \frac{\sigma_x - \sigma_y}{2} = -29 \text{ MPa},$$

$$\sin 2\theta = \sin(-30^\circ) = -0.5, \quad \cos 2\theta = \cos(-30^\circ) = 0.8660.$$

Substituting into the transformation equations, we get

$$\begin{aligned} \sigma_{x_1} &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta = \\ &= -17 \text{ MPa} + (-29 \text{ MPa})(0.8661) + (-19 \text{ MPa})(-0.5) = -32.6 \text{ MPa}, \end{aligned}$$

$$\begin{aligned} \tau_{x_1 y_1} &= -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta = \\ &= -(-29 \text{ MPa})(-0.5) + (-19 \text{ MPa})(0.8660) = -31.0 \text{ MPa}. \end{aligned}$$

Also, the normal stress acting on the  $y_1$  face (Eq. (3.10)) is

$$\begin{aligned} \sigma_{y_1} &= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta = \\ &= -17 \text{ MPa} - (-29 \text{ MPa})(0.8661) - (-19 \text{ MPa})(-0.5) = -1.4 \text{ MPa}. \end{aligned}$$

To check the results, we note that  $\sigma_{x_1} + \sigma_{y_1} = \sigma_x + \sigma_y$ .

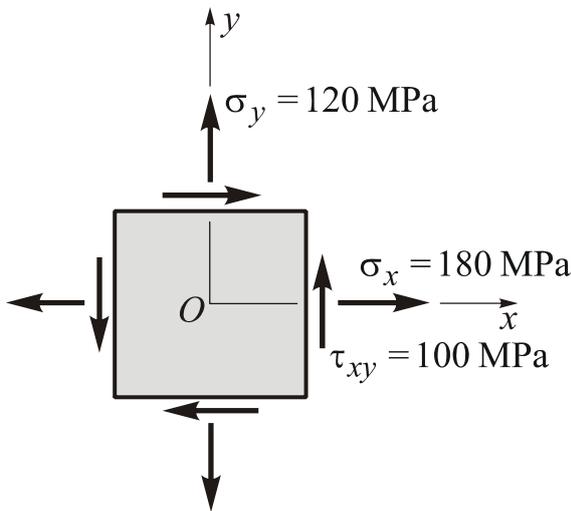
The stresses acting on the inclined element are shown in Fig. b, where the arrows indicate the true directions of the stresses.

**Note.** Both stress elements shown in the figure represent the same state of stress.



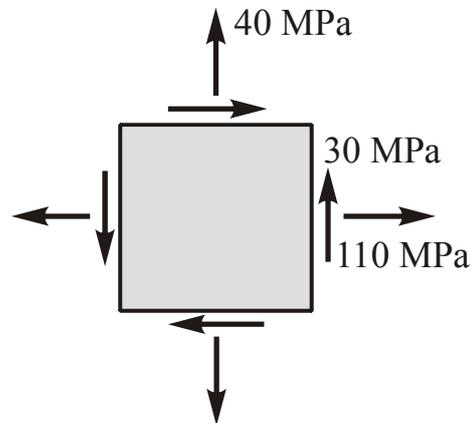
## PROBLEMS

**Problem 3.1** An element in plane stress is subjected to stresses  $\sigma_x = 180 \text{ MPa}$ ,  $\sigma_y = 120 \text{ MPa}$ , and  $\tau_{xy} = 100 \text{ MPa}$ , as shown in the figure. Determine the stresses acting on an element oriented at an angle  $\theta = 60^\circ$  from the  $x$  axis, where the angle  $\theta$  is positive when counterclockwise. Show these stresses on a sketch of an element oriented at the angle  $\theta$ .

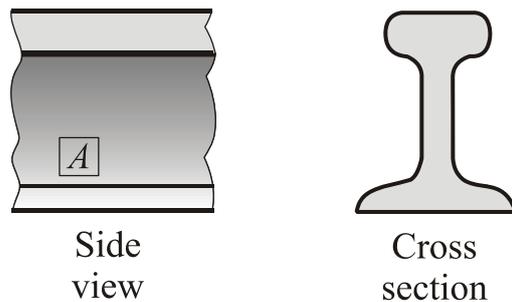


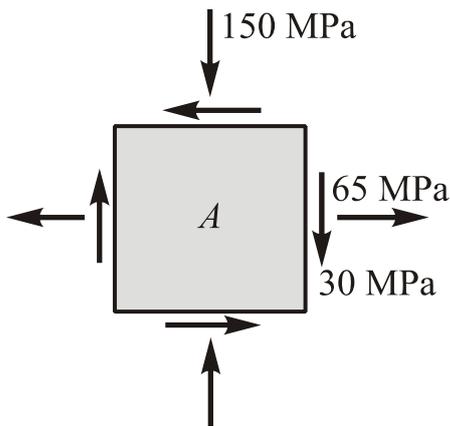
**Problem 3.2** An element in plane stress is subjected to stresses  $\sigma_x = 110 \text{ MPa}$ ,  $\sigma_y = 40 \text{ MPa}$ , and  $\tau_{xy} = 30 \text{ MPa}$ , as shown in the figure. Determine the stresses acting on an element oriented at an angle  $\theta = -30^\circ$  from the  $x$  axis, where the angle  $\theta$  is

negative when clockwise. Show these stresses on a sketch of an element oriented at the angle  $\theta$ .

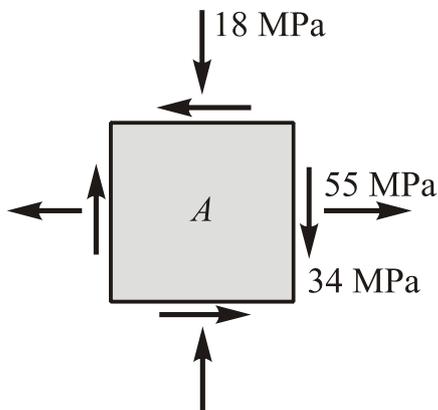
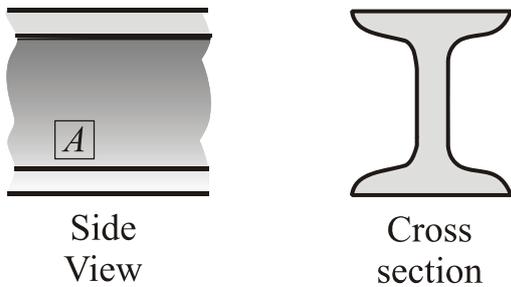


**Problem 3.3** The stresses at point  $A$  in the web of a train rail are equal to be  $65 \text{ MPa}$  (tension) in the horizontal direction and  $150 \text{ MPa}$  (compression) in the vertical direction (see figure). Also, the shear stresses are  $30 \text{ MPa}$  acting in the directions shown. Determine the stresses acting on an element oriented at a counterclockwise angle of  $60^\circ$  from the horizontal.

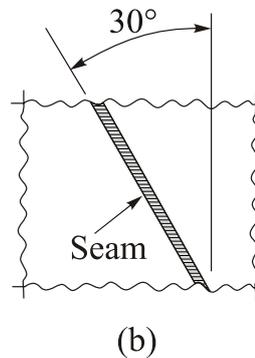
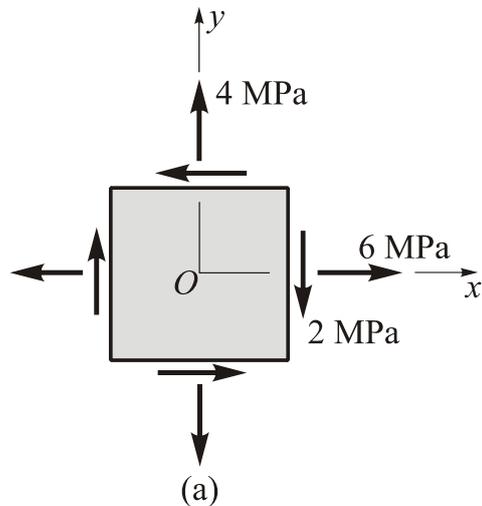




**Problem 3.4** The stresses at point A in the web of a wide-flange beam are found to be 55 MPa (tension) in the horizontal direction and 18 MPa compression in the vertical direction (see figure). Also, the shear stresses are 34 MPa acting in the directions shown. Determine the stresses acting on an element oriented at a counterclockwise angle of  $50^\circ$  to the horizontal.

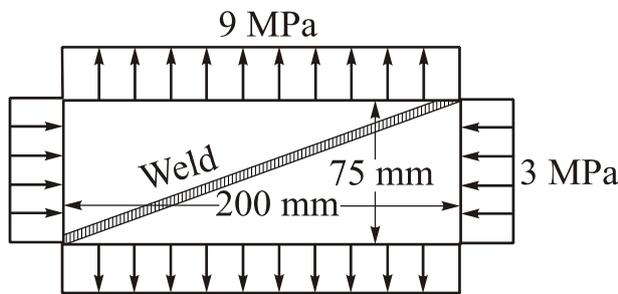


**Problem 3.5** The polyethylene liner of a settling pond is subjected to stresses  $\sigma_x = 6$  MPa,  $\sigma_y = 4$  MPa, and  $\tau_{xy} = -2$  MPa, as shown by the plane-stress element in part (a) of the figure. Determine the normal and shear stresses acting on a seam inclined at an angle of  $30^\circ$  to the element, as shown in part (b) of the figure.

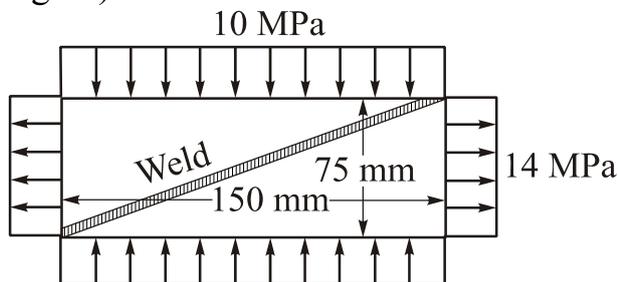


**Problem 3.6** A thin rectangular plate of dimensions  $75 \text{ mm} \times 200 \text{ mm}$  is formed by welding two triangular plates (see figure). The plate is subjected to a compressive stress of 3.0 MPa in the long direction and a tensile stress of 9.0 MPa in the short direction. Determine the normal stress  $\sigma_w$  acting

perpendicular to the line of the weld and the shear stress  $\tau_w$  acting parallel to the weld.

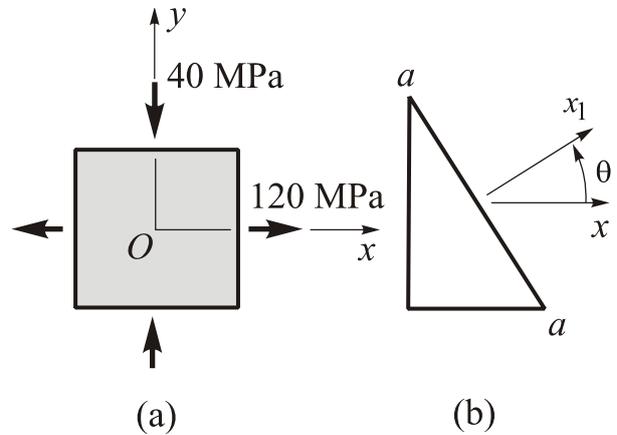


**Problem 3.7** Solve the preceding problem for a plate of dimensions  $75 \text{ mm} \times 150 \text{ mm}$  subjected to a tensile stress of  $14 \text{ MPa}$  in the long direction and a compressive stress of  $10 \text{ MPa}$  in the short direction (see figure).

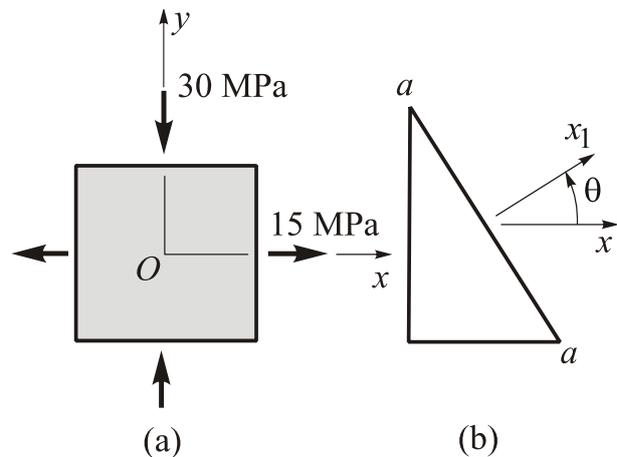


**Problem 3.8** At a point on the surface of a machine the material is in biaxial stress with  $\sigma_x = 120 \text{ MPa}$  and  $\sigma_y = -40 \text{ MPa}$ , as shown in part (a) of the figure. Part (b) of the figure shows an inclined plane  $a-a$  through the same point in the material but oriented at an angle  $\theta$ . Determine all values of the angle  $\theta$  such that no normal stresses act on plane  $a-a$ . For each angle  $\theta$ , sketch a stress element having plane  $a-a$  as one

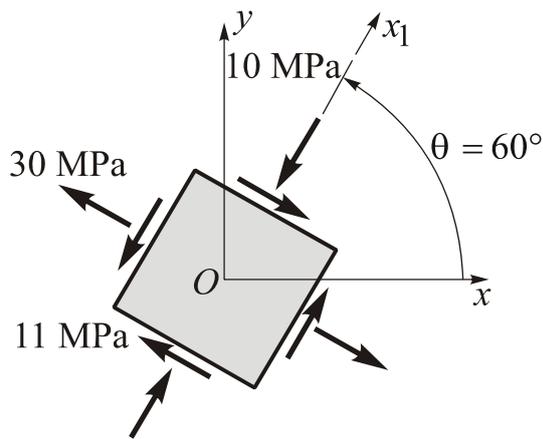
of its sides and show all stresses acting on the element.



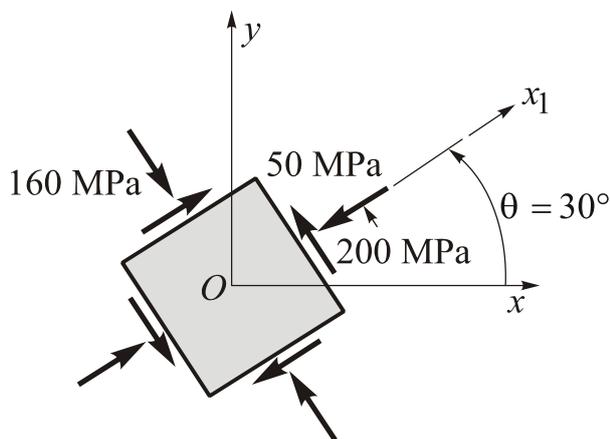
**Problem 3.9** Solve the preceding problem for  $\sigma_x = 15 \text{ MPa}$  and  $\sigma_y = -30 \text{ MPa}$  (see figure).



**Problem 3.10** An element in plane stress from the frame of a machine is oriented at a known angle  $\theta = 60^\circ$  (see figure). On this inclined element, the normal and shear stresses have the magnitudes and directions shown in the figure. Determine the normal and shear stresses acting on an element whose sides are parallel to the  $xy$  axes; that is, determine  $\sigma_x$ ,  $\sigma_y$ , and  $\tau_{xy}$ .

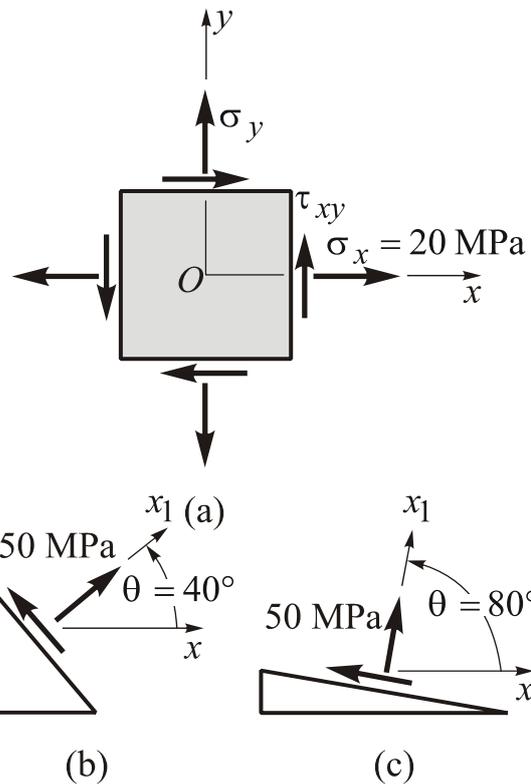


**Problem 3.11** An element in plane stress from the frame of a car is oriented at a known angle  $\theta$  (see Fig.). On this inclined element, the normal and shear stresses have the magnitudes and directions shown in the Fig. Determine the normal and shear stresses acting on an element whose sides are parallel to the  $xy$  axes; that is, determine  $\sigma_x$ ,  $\sigma_y$ , and  $\tau_{xy}$ .

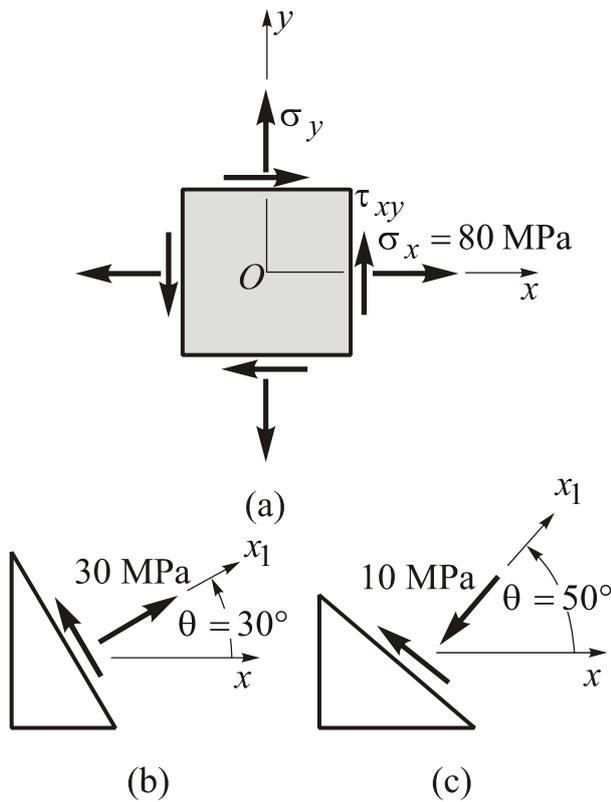


**Problem 3.12** A thin plate in plane stress is subjected to normal stresses  $\sigma_x$  and  $\sigma_y$  and a shear stress  $\tau_{xy}$ , as shown in part (a) of the figure. At angles  $\theta = 40^\circ$  and  $80^\circ$  from the  $x$  axis

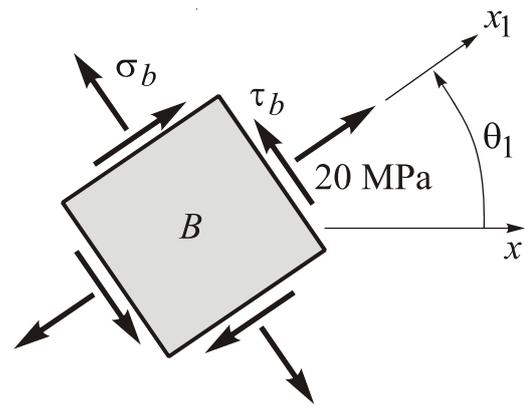
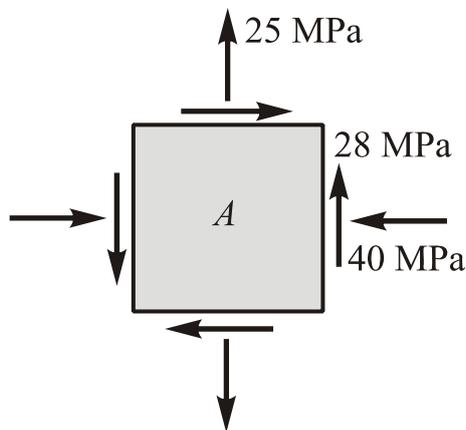
the normal stress is 50 MPa (tension) (see parts (b) and (c) of the figure). If the stress  $\sigma_x$  equals 20 MPa (tension), what are the stresses  $\sigma_y$  and  $\tau_{xy}$ ?



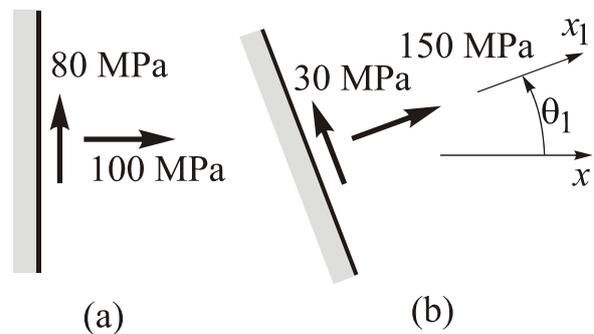
**Problem 3.13** The surface of an airplane wing is subjected to plane stress with normal stresses  $\sigma_x$  and  $\sigma_y$  and shear stress  $\tau_{xy}$ , as shown in part (a) of the figure. At an angle  $\theta = 30^\circ$  from the  $x$  axis the normal stress is 30 MPa (tension), and at an angle  $\theta = 50^\circ$  it is 10 MPa (compression) (see parts (b) and (c) of the figure). If the stress  $\sigma_x$  equals 80 MPa (tension), what are the stresses  $\sigma_y$  and  $\tau_{xy}$ ?



**Problem 3.14** At a point in a structure subjected to plane stress, the stresses have the magnitudes and directions shown acting on element *A* in the first part of the figure. Element *B*, located at the same point in the structure, is oriented at an angle  $\theta_1$  (between zero and  $90^\circ$ ) such that the stresses are as shown in the second part of the figure. Calculate the normal stress  $\sigma_b$ , the shear stress  $\tau_b$  and the angle  $\theta_1$ .

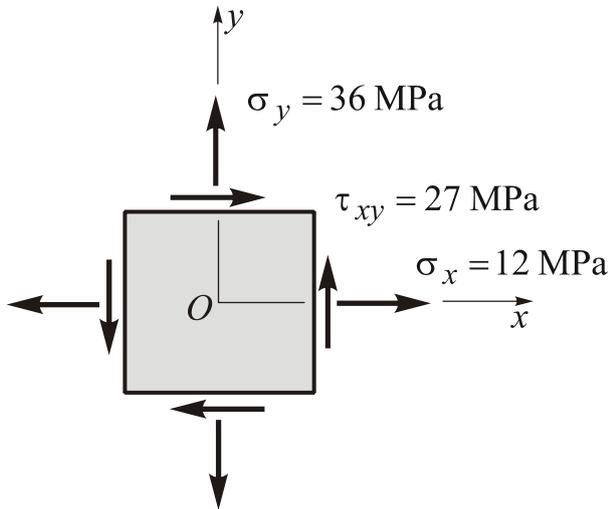


**Problem 3.15** At a given point of the structural element the material is subjected to plane stress such that on a vertical plane through the point, the normal stress is  $100 \text{ MPa}$  (tension) and the shear stress is  $80 \text{ MPa}$  counterclockwise (see part (a) of the figure), and on a plane oriented at an angle  $\theta_1$  the normal stress is  $150 \text{ MPa}$  (tension) and the shear stress is  $30 \text{ MPa}$  counterclockwise (see part (b) of the figure). What is the angle  $\theta_1$ ?

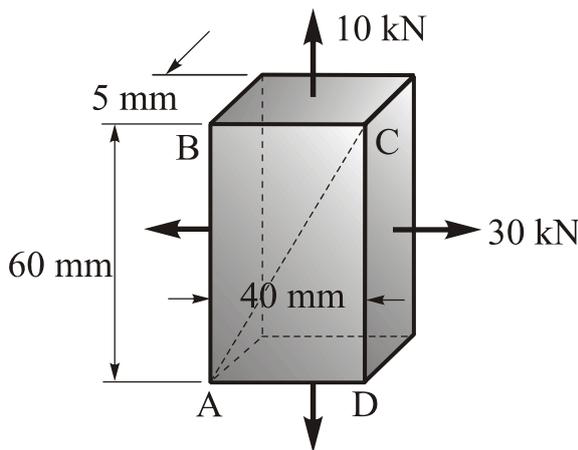


**Problem 3.16** The stresses on the surface of a crane hook are  $\sigma_x = 12 \text{ MPa}$ ,  $\sigma_y = 36 \text{ MPa}$ , and  $\tau_{xy} = 27 \text{ MPa}$ , as shown by the stress element in the figure. For what angles  $\theta$  between  $0$  and  $90^\circ$  (counterclockwise) is the normal stress equal numerically to twice the shear stress? For each such

angle, sketch a properly oriented stress element and show all stresses acting on the element.

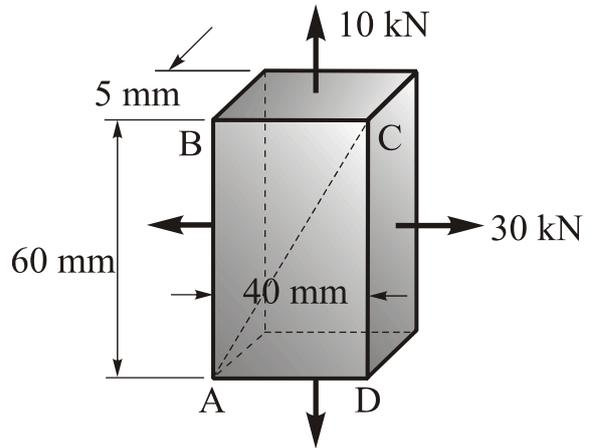


**Problem 3.17** A 60-mm by 40-mm plate of 5-mm thickness is subjected to uniformly distributed biaxial tensile forces (see figure). What normal and shearing stresses exist along diagonal  $AC$ ? Use the equilibrium equation applied to the wedge-shaped half  $ABC$  of the plate.

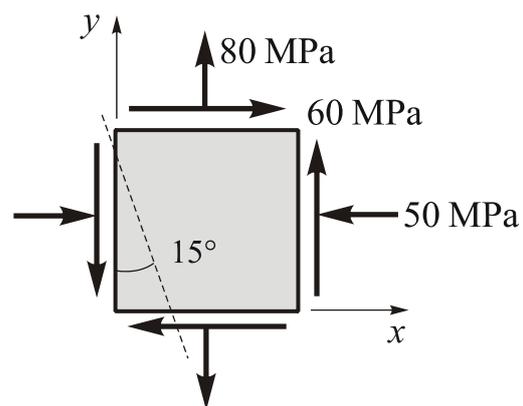


**Problem 3.18** A 60-mm by 40-mm plate of 5-mm thickness is subjected to uniformly distributed

biaxial tensile forces (see figure). What normal and shearing stresses exist along diagonal  $AC$ ? Use in solution the Eqs. (3.8, 3.9, 3.10) and compare with the result of Problem 3.17.

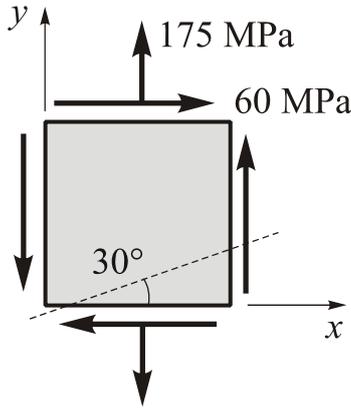


**Problem 3.19** The state of stress at the point in a loaded body is represented in the figure. Determine the normal and shearing stresses acting on the indicated inclined plane. Use an approach based upon the equilibrium equations applied to the wedge-shaped element.

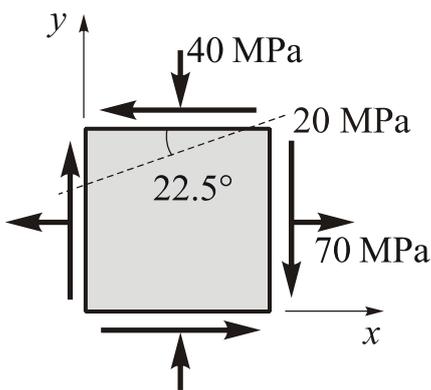


**Problem 3.20** The state of stress at the point in a loaded body is represented in the figure. Determine the

normal and shearing stresses acting on the indicated inclined plane. Use an approach based upon the equilibrium equations applied to the wedge-shaped element shown.

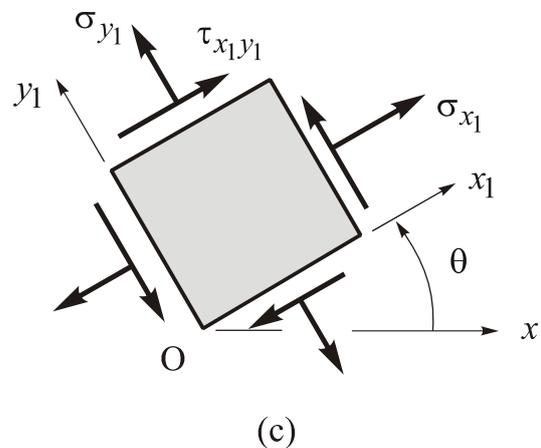
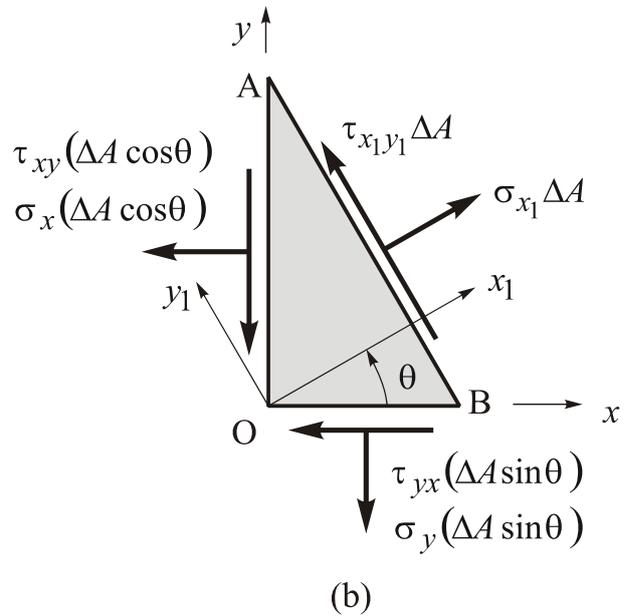
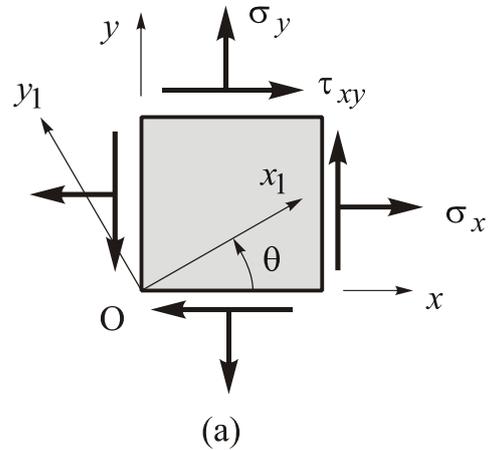


**Problem 3.21** The state of stress at the point in a loaded body is represented in the figure. Determine the normal and shearing stresses acting on the indicated inclined plane. Use an approach based upon the equilibrium equations applied to the wedge-shaped element shown.

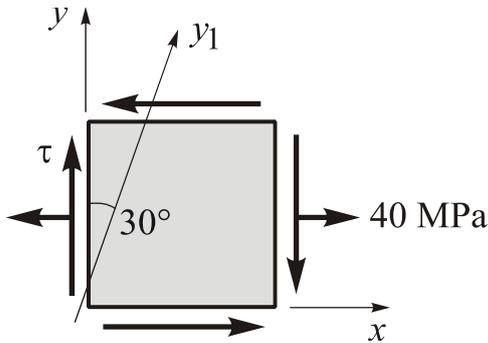


**Problem 3.22** The stresses at a point (see figure (a)) are  $\sigma_x = \sigma_y = 0$  and  $\tau_{xy} = 100 \text{ MPa}$  (pure shear).

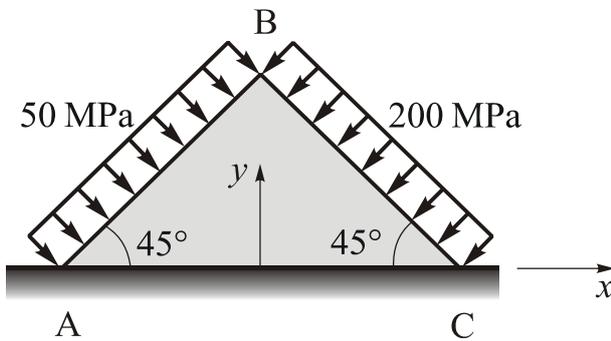
Determine the stresses on all sides of an element rotated through an angle  $\theta = 25^\circ$ .



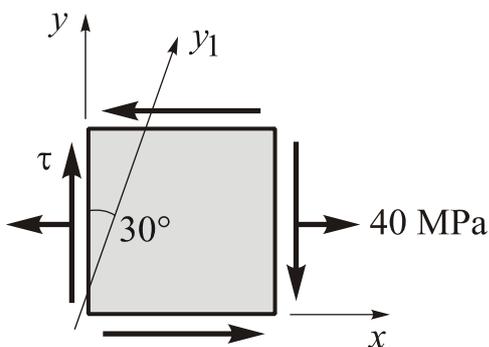
**Problem 3.23** At a point in a loaded structural element, the stresses are as shown in the figure. The normal stress at the point on the indicated plane is 40 MPa (tension). What is the magnitude of the shearing stress  $\tau$ ?



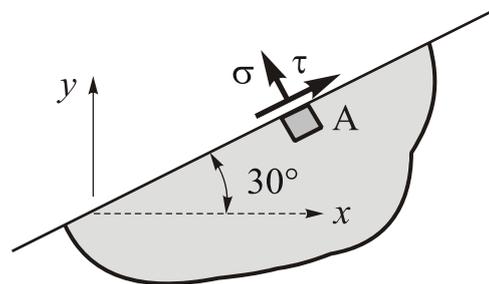
**Problem 3.24** A triangular plate is subjected to stresses as shown in the figure. Determine  $\sigma_x$ ,  $\sigma_y$ , and  $\tau_{xy}$  and sketch the results on a properly oriented element.



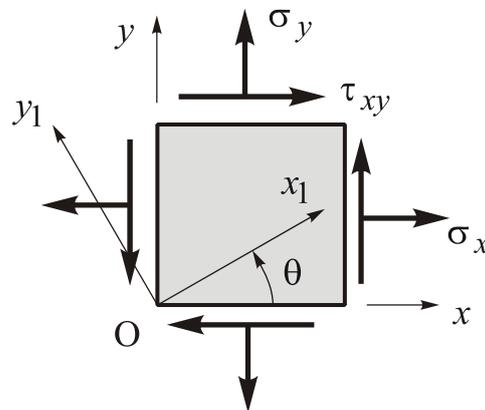
**Problem 3.25** Calculate the normal and shearing stresses acting on the plane indicated in the figure for  $\tau = 30$  MPa.



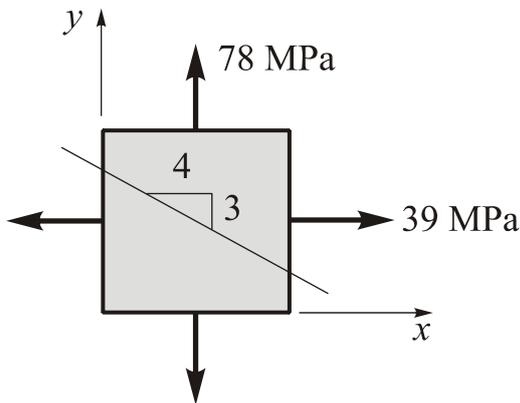
**Problem 3.26** At a critical point  $A$  in the loaded element (see figure), the stresses on the inclined plane are  $\sigma = 28$  MPa and  $\tau = 12$  MPa, and the normal stress on the  $y$  plane is zero. Calculate the normal and shear stresses on the  $x$  plane through the point. Show the results on a properly oriented element.



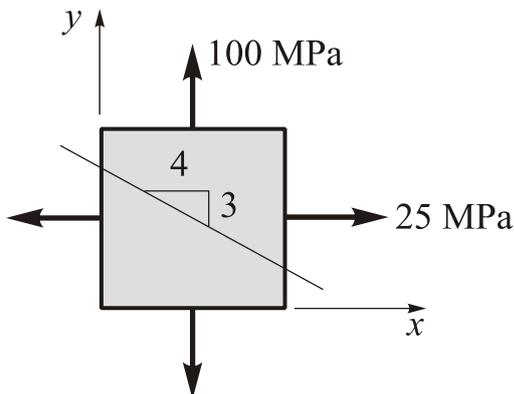
**Problem 3.27** At a given point in a loaded machine part (see figure), the stresses are  $\sigma_x = 40$  MPa,  $\sigma_y = -25$  MPa and  $\tau_{xy} = 0$ . Determine the normal and shear stresses on the plane whose normals are at angles of  $-30^\circ$  and  $120^\circ$  with the  $x$  axis. Sketch the results on properly oriented elements.



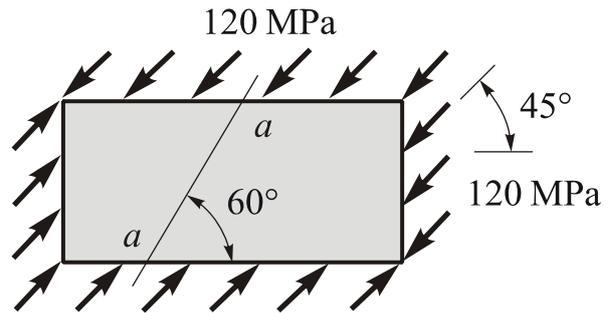
**Problem 3.28** The stresses at a point of a boiler are as shown in the element represented in the figure. Calculate the normal and shear stresses at the point on the indicated inclined plane.



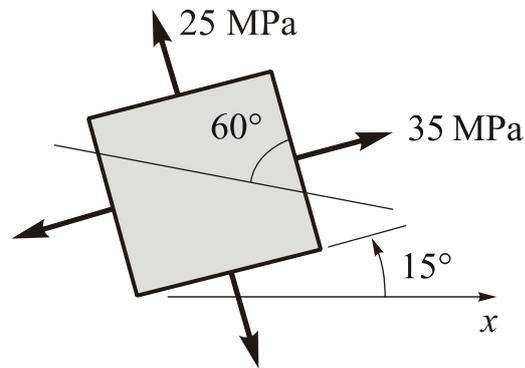
**Problem 3.29** The stresses at a point in the pressure vessel are as shown in the element represented in the figure. Calculate the normal and shear stresses at the point on the indicated inclined plane.



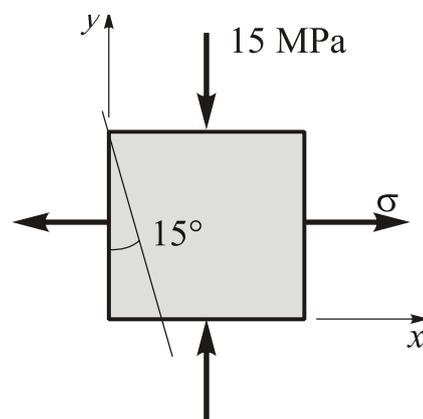
**Problem 3.30** The stresses acting uniformly at the edges of a thick, rectangular plate are shown in the figure. Determine the stress components on planes parallel and perpendicular to  $a-a$ . Show the results on a properly oriented element.



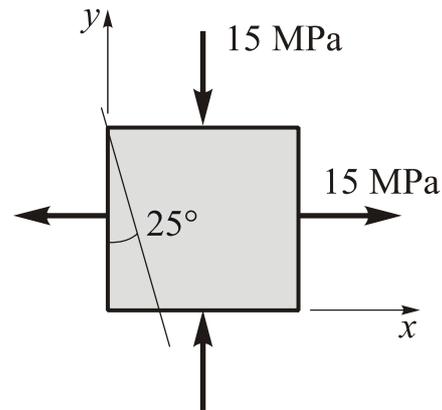
**Problem 3.31** The figure represents the state of stress at a point in a structural member. Calculate the normal and shear stresses at the point on the indicated inclined plane. Sketch the results on a properly oriented element.



**Problem 3.32** At a point in a loaded member, the stresses are shown in the figure. Determine the allowable value of  $\sigma$  if the normal and shearing stresses acting simultaneously in the indicated inclined plane are limited to 35 MPa and 20 MPa, respectively.



**Problem 3.33** Calculate the normal and shear stresses acting on the inclined plane in the figure. for  $\sigma_x = 15 \text{ MPa}$  and  $\sigma_y = -15 \text{ MPa}$ .



### 3.3 Principal Stresses and Maximum Shear Stresses

The transformation equations for plane stress show that the normal stresses  $\sigma_{x_1}$  and the shear stresses  $\tau_{x_1y_1}$  vary continuously as the axes are rotated through the angle  $\theta$ . This variation is pictured in Fig. 3.9 for a particular combination of stresses. From the figure, we see that both the normal and shear stresses reach maximum and minimum values at  $90^\circ$  intervals. These *maximum and minimum values are usually needed for design purposes*. For instance, fatigue failures of structures such as machines and aircraft are often associated with the maximum stresses, and hence their magnitudes and orientations should be determined as part of the design process.

The determination of principal stresses is an example of a type of mathematical analysis known as *eigenvalue problem* in matrix algebra. The stress-transformation equations and the concept of principal stresses are due to the French mathematicians A. L. Cauchy (1789-1857) and Barre de Saint-Venant (1797-1886) and to the Scottish scientist and engineer W. J. M. Rankine (1820-1872).

#### 3.3.1 Principal Stresses

The maximum and minimum normal stresses, called the *principal stresses*, can be found from the transformation equation for the normal stress  $\sigma_{x_1}$  (Eq. 3.8). By taking the derivative of  $\sigma_{x_1}$  with respect to  $\theta$  and setting it equal to zero, we obtain an equation from which we can find the values of  $\theta$  at which  $\sigma_{x_1}$  is a maximum or minimum. The equation for the derivative is

$$\frac{d\sigma_{x_1}}{d\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0, \quad (3.18)$$

from which we get

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}. \quad (3.19)$$

The subscript  $p$  indicates that the angle  $\theta_p$  defines the orientation of the *principal planes*, i.e. the planes, on which the principal stresses act.

Two values of the angle  $2\theta_p$  in the range from 0 to  $360^\circ$  can be obtained from Eq. (3.19). These values differ by  $180^\circ$ , with one value between 0 and  $180^\circ$  and the other between  $180^\circ$  and  $360^\circ$ . Therefore, the angle  $\theta_p$  has two values that differ by  $90^\circ$ , one value between 0 and  $90^\circ$  and the other between  $90^\circ$  and  $180^\circ$ . The two values of  $\theta_p$  are known as the *principal angles*. For one of these angles, the normal stress  $\sigma_{x_1}$  is a *maximum principal stress*; for the other, it is a *minimum principal stress*. Because the principal angles differ by  $90^\circ$ , we see that *the principal stresses occur on mutually perpendicular planes*.

The principal stresses can be calculated by substituting each of the two values of  $\theta_p$  into the first stress-transformation equation (Eq. 3.8) and solving for  $\sigma_{x_1}$ . By determining the principal stresses in this manner, we not only obtain the values of the principal stresses but *we also learn which principal stress is associated with which principal angle*.

Let us obtain the formulas for the principal stresses, using right triangle in Fig. 3.14, constructed from Eq. (3.19). The hypotenuse of the triangle, obtained from the Pythagorean theorem, is

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}. \quad (3.20)$$

The quantity  $R$  is always a positive number and, like the other two sides of the triangle, has units of stress. From the triangle we obtain two additional relations:

$$\cos 2\theta_p = \frac{\sigma_x - \sigma_y}{2R}, \quad (3.21)$$

$$\sin 2\theta_p = \frac{\tau_{xy}}{R}. \quad (3.22)$$

Now we substitute these expressions for  $\cos 2\theta_p$  and  $\sin 2\theta_p$  into Eq. (3.8) and obtain the *algebraically larger of the two principal stresses*, denoted by  $\sigma_1$ :

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}. \quad (3.23)$$

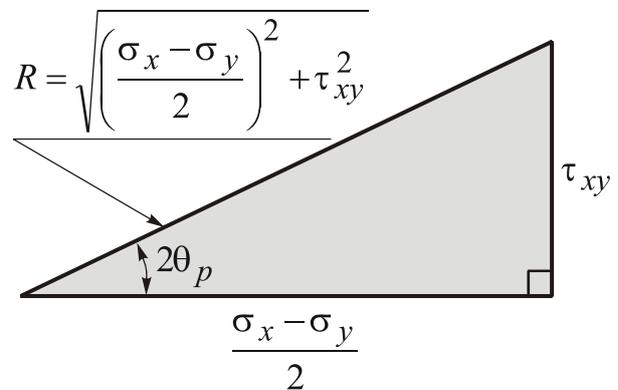


Fig. 3.14 Geometric analogue of Eq. (3.19)

The smaller of the principal stresses, denoted by  $\sigma_2$ , may be found from the condition that the sum of the normal stresses on perpendicular planes is constant (see Eq. 3.11):

$$\sigma_1 + \sigma_2 = \sigma_x + \sigma_y. \quad (3.24)$$

Substituting the expression for  $\sigma_1$  into Eq. (3.24) and solving for  $\sigma_2$ , we get

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}. \quad (3.25)$$

The formulas for  $\sigma_1$  and  $\sigma_2$  can be combined into a single formula for the *principal stresses*:

$$\sigma_{\max, \min} = \sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}. \quad (3.26)$$

**Note. The plus sign gives the algebraically larger principal stress and the minus sign gives the algebraically smaller principal stress.**

Let us now find two angles defining the principal planes as  $\theta_{p_1}$  and  $\theta_{p_2}$ , corresponding to the principal stresses  $\sigma_1$  and  $\sigma_2$ , respectively. Both angles can be determined from the equation for  $\tan 2\theta_p$  (Eq. 3.19). To correlate the principal angles and principal stresses we will use Eqs. (3.21) and (3.22) to find  $\theta_p$  since the only angle that satisfies *both* of those equations is  $\theta_{p_1}$ . Thus, we can rewrite those equations as follows:

$$\cos 2\theta_{p_1} = \frac{\sigma_x - \sigma_y}{2R}, \quad (3.27)$$

$$\sin 2\theta_{p_1} = \frac{\tau_{xy}}{R}. \quad (3.28)$$

Only one angle exists between  $0$  and  $360^\circ$  that satisfies both of these equations. Thus, the value of  $\theta_{p_1}$  can be determined uniquely from Eqs. (3.27) and (3.28).

The angle  $\theta_{p_2}$ , corresponding to  $\sigma_2$ , defines a plane that is perpendicular to the plane defined by  $\theta_{p_1}$ . Therefore,  $\theta_{p_2}$  can be taken as  $90^\circ$  larger or  $90^\circ$  smaller than  $\theta_{p_1}$ .

It is very important to know the value of shear stresses acting at principal planes. For this we will use the transformation equation for the shear stresses (Eq. (3.9)). If we set the shear stress  $\tau_{x_1y_1}$  equal to zero, we get an equation that is the same as Eq. (3.18). It means that the *angles to the planes of zero shear stress*

are the same as the angles to the principal planes. Thus, the shear stresses are zero on the principal planes.

The principal planes for elements in *uniaxial stress* and *biaxial stress* are the  $x$  and  $y$  planes themselves (Fig. 3.15), because  $\tan 2\theta_p = 0$  (see Eq. 3.19) and the two values of  $\theta_p$  are  $0$  and  $90^\circ$ . We also know that the  $x$  and  $y$  planes are the principal planes from the fact that the shear stresses are zero on those planes.

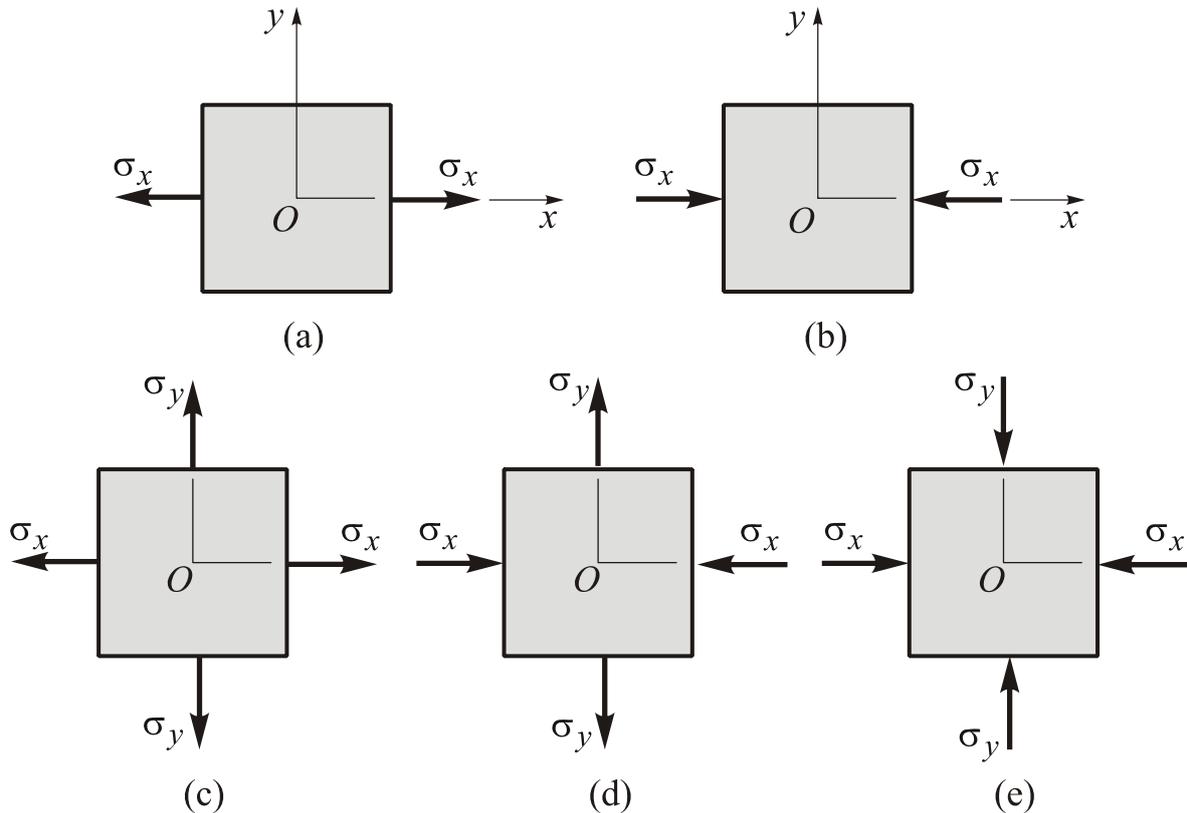


Fig. 3.15 Elements in uniaxial (a) and (b) and biaxial (c), (d), (e) stress state:

- (a)  $\sigma_x = 80 \text{ MPa} = \sigma_1$ ,  $\sigma_y = 0 = \sigma_{2(3)}$ ,  $\sigma_z = 0 = \sigma_{3(2)}$ ; (b)  $\sigma_x = -80 \text{ MPa} = \sigma_3$ ,  
 $\sigma_y = 0 = \sigma_{1(2)}$ ,  $\sigma_z = 0 = \sigma_{2(1)}$ ;  
(c)  $\sigma_x = 60 \text{ MPa} = \sigma_1$ ,  $\sigma_y = 25 \text{ MPa} = \sigma_2$ ,  $\sigma_z = 0 = \sigma_3$ ; (d)  $\sigma_x = -60 \text{ MPa} = \sigma_3$ ,  
 $\sigma_y = 25 \text{ MPa} = \sigma_1$ ,  $\sigma_z = 0 = \sigma_2$ ;  
(e)  $\sigma_x = -60 \text{ MPa} = \sigma_3$ ,  $\sigma_y = -25 \text{ MPa} = \sigma_2$ ,  $\sigma_z = 0 = \sigma_1$

For an element in *pure shear* (Fig. 3.16a), the principal planes are oriented at  $45^\circ$  to the  $x$  axis (Fig. 3.16b), because  $\tan 2\theta_p$  is infinite and the two values of  $\theta_p$  are  $45^\circ$  and  $135^\circ$ . If  $\tau_{xy}$  is positive, the principal stresses are  $\sigma_1 = \tau_{xy}$  and  $\sigma_2 = -\tau_{xy}$ .

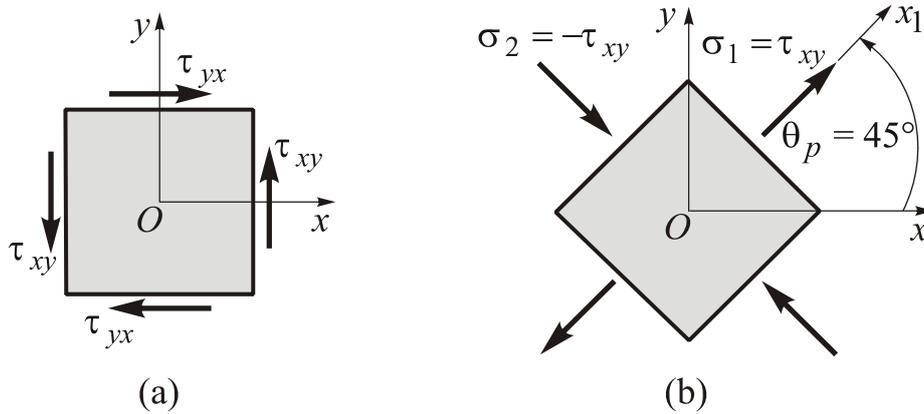


Fig. 3.16 Element in pure shear

Two principal stresses determined from Eq. (3.26) are called the *in-plane principal stresses*. Really any stress element is three-dimensional (Fig. 3.17a) and has three (not two) principal stresses acting on three mutually perpendicular planes. By making a more complete three-dimensional analysis, it can be shown that the three principal planes for a plane-stress element are the two principal planes already described plus the  $z$  face of the element. These principal planes are shown in Fig. 3.17b, where a stress element has been oriented at the principal angle  $\theta_{p1}$  which corresponds to the principal stress  $\sigma_1$ . The principal stresses  $\sigma_1$  and  $\sigma_2$  are given by Eq. (3.26), and the third principal stress ( $\sigma_3$ ) equals zero. By definition,  $\sigma_1$  is algebraically the largest and  $\sigma_3$  is algebraically the smallest one.

**Note.** There are no shear stresses on any of the principal planes.

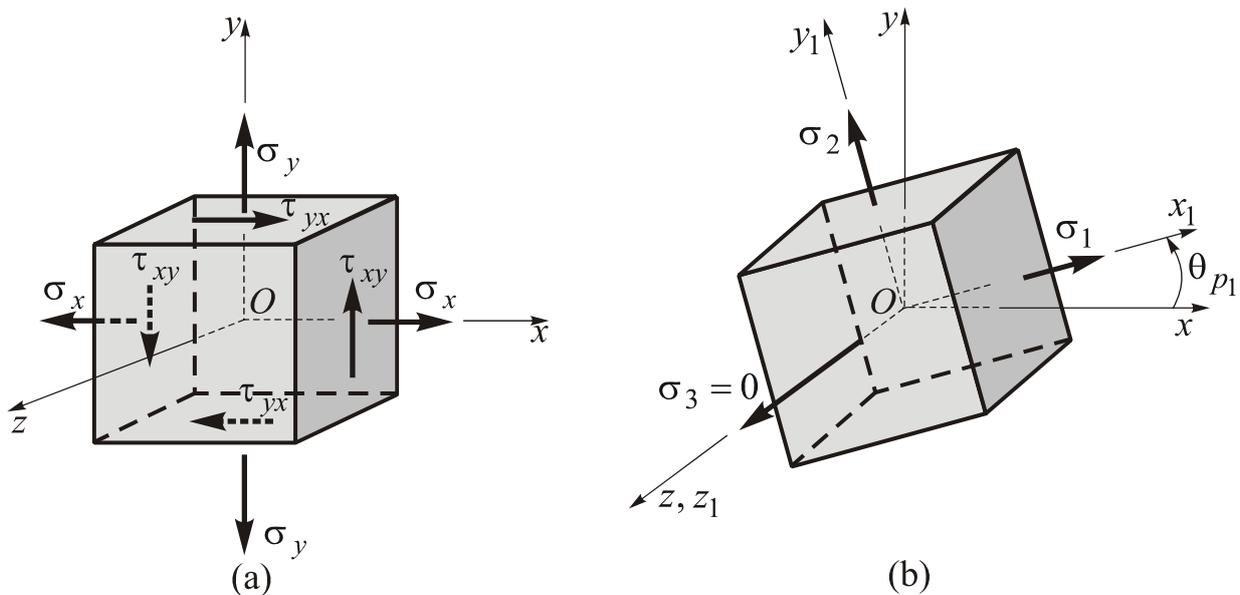


Fig. 3.17 Elements in plane stress: (a) original element, and (b) element oriented to the three principal planes and three principal stresses

### 3.3.2 Maximum Shear Stresses

Now we consider the determination of the maximum shear stresses and the planes on which they act. The shear stresses  $\tau_{x_1y_1}$  acting on inclined planes are given by the second transformation equation (Eq. 3.9). Equating the derivative of  $\tau_{x_1y_1}$  with respect to  $\theta$  to zero, we obtain

$$\frac{d\tau_{x_1y_1}}{d\theta} = -(\sigma_x - \sigma_y)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0, \quad (3.29)$$

from which

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}. \quad (3.30)$$

The subscript  $s$  indicates that the angle  $\theta_s$  defines the orientation of the planes of maximum positive and negative shear stresses. Equation (3.30) yields one value of  $\theta_s$  between  $0$  and  $90^\circ$  and another between  $90^\circ$  and  $180^\circ$ . These two values differ by  $90^\circ$ , and therefore the maximum shear stresses occur on perpendicular planes. Because shear stresses on perpendicular planes are equal in absolute value, the maximum positive and negative shear stresses differ only in sign.

Comparing Eq. (3.30) for  $\theta_s$  with Eq. (3.19) for  $\theta_p$  shows that

$$\tan 2\theta_s = -\frac{1}{\tan 2\theta_p} = -\cot 2\theta_p. \quad (3.31)$$

This equation is the relationship between the angles  $\theta_s$ , and  $\theta_p$ . Let us rewrite this equation in the form

$$\frac{\sin 2\theta_s}{\cos 2\theta_s} + \frac{\cos 2\theta_p}{\sin 2\theta_p} = 0, \quad (3.32)$$

or

$$\sin 2\theta_s \sin 2\theta_p + \cos 2\theta_s \cos 2\theta_p = 0. \quad (3.33)$$

Eq. (3.33) is equivalent to the following expression:

$$\cos(2\theta_s - 2\theta_p) = 0.$$

Therefore,

$$2\theta_s - 2\theta_p = \pm 90^\circ,$$

and

$$\theta_s = \theta_p \pm 45^\circ. \quad (3.34)$$

**Note. Eq. (3.34) shows that the planes of maximum shear stress occur at  $45^\circ$  to the principal planes.**

The plane of the maximum positive shear stress  $\tau_{\max}$  is defined by the angle  $\theta_{s_1}$ , for which the following equations apply:

$$\cos 2\theta_{s_1} = \frac{\tau_{xy}}{R}, \quad (3.35)$$

$$\sin 2\theta_{s_1} = -\frac{\sigma_x - \sigma_y}{2R}, \quad (3.36)$$

in which  $R$  is given by Eq. (3.20). Also, the angle  $\theta_{s_1}$  is related to the angle  $\theta_{p_1}$  (see Eqs. (3.27) and (3.28)) as follows:

$$\theta_{s_1} = \theta_{p_1} - 45^\circ. \quad (3.37)$$

Corresponding maximum shear stress is obtained by substituting the expressions for  $\cos 2\theta_{s_1}$  and  $\sin 2\theta_{s_1}$  into the second transformation equation (Eq. 3.9), yielding

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}. \quad (3.38)$$

The maximum negative shear stress has the same magnitude but opposite sign.

Another expression for the maximum shear stress  $\tau_{\max}$  can be obtained from the principal stresses  $\sigma_1$  and  $\sigma_2$ , both of which are given by Eq. (3.26). Subtracting the expression for  $\sigma_2$  from that for  $\sigma_1$  and then comparing with Eq. (3.38), we see that

$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2}. \quad (3.39)$$

**Note. Maximum shear stress is equal to one-half the difference of the principal stresses.**

The planes of maximum shear stress  $\tau_{\max}$  also contain normal stresses. The normal stress acting on the planes of maximum positive shear stress can be determined by substituting the expressions for the angle  $\theta_{s_1}$  (Eqs. (3.35) and (3.36)) into the equation for  $\sigma_{x_1}$  (Eq. 3.8). The resulting stress is equal to the *average* of the normal stresses on the  $x$  and  $y$  planes:

$$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2}. \quad (3.40)$$

This same normal stress acts on the planes of maximum negative shear stress.

In the particular cases of uniaxial stress and biaxial stress (Fig. 3.15), the planes of maximum shear stress occur at  $45^\circ$  to the  $x$  and  $y$  axes. In the case of pure shear (Fig. 3.16), the maximum shear stresses occur on the  $x$  and  $y$  planes.

The analysis of shear stresses has dealt only with the stresses acting in the  $xy$  plane, i.e. in-plane shear stress. The maximum in-plane shear stresses were found on an element obtained by rotating the  $x, y, z$  axes (Fig. 3.17a) about the  $z$  axis through an angle of  $45^\circ$  to the principal planes. The principal planes for the element of Fig. 3.17a are shown in Fig. 3.17b.

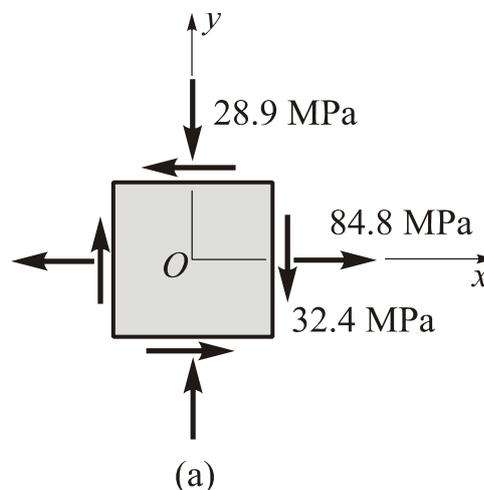
We can also obtain maximum shear stresses by  $45^\circ$  rotations about the other two principal axes (the  $x_1$  and  $y_1$  axes in Fig. 3.17b). As a result, we obtain three sets of maximum positive and maximum negative shear stresses (compare with Eq. (3.39)).

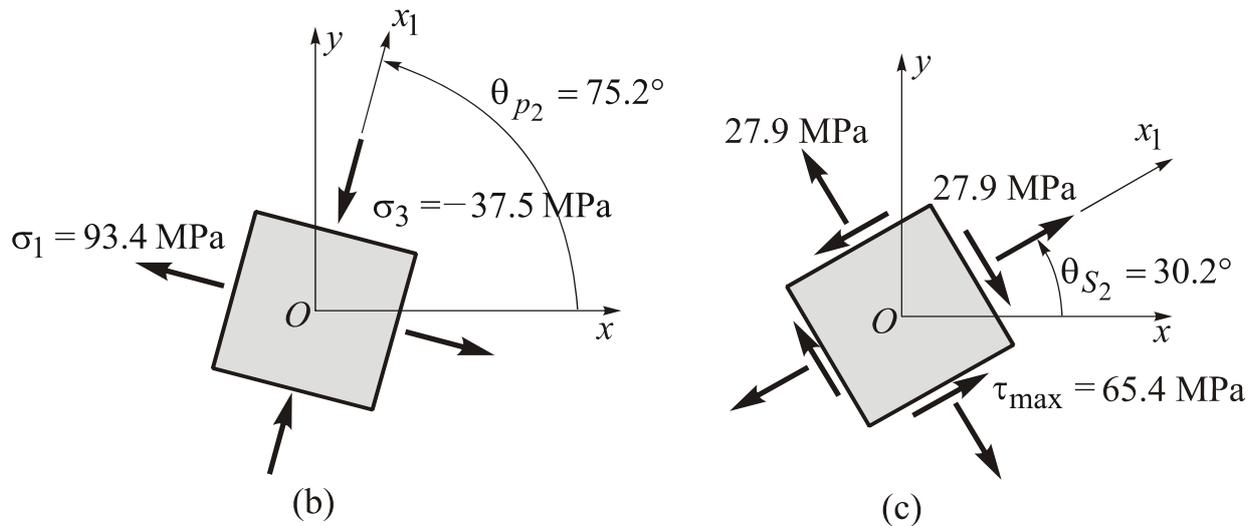


## EXAMPLES

### Example 3.5

An element in plane stress is subjected to stresses  $\sigma_x = 84.8$  MPa,  $\sigma_y = -28.9$  MPa, and  $\tau_{xy} = -32.4$  MPa, as shown in Fig. a. (1) Determine the principal stresses and show them on a sketch of a properly oriented element; (2) Determine the maximum shear stresses and show them on a sketch of a properly oriented element.





(a) Element in plane stress; (b) principal stresses; and (c) maximum shear stresses

**Solution** (1) *Calculation of principal stresses.* The principal angles  $\theta_p$  that locate the principal planes can be obtained from Eq. (3.19):

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(-32.4 \text{ MPa})}{84.8 \text{ MPa} - (-28.9 \text{ MPa})} = -0.5697.$$

Solving for the angles, we get the following two sets of values:

$$2\theta_p = 150.3^\circ \text{ and } \theta_p = 75.2^\circ,$$

$$2\theta_p = 330.3^\circ \text{ and } \theta_p = 165.2^\circ.$$

The principal stresses may be obtained by substituting the two values of  $2\theta_p$  into the transformation equation for  $\sigma_{x_1}$  (Eq. (3.8)). Determine preliminary the following quantities:

$$A = \frac{\sigma_x + \sigma_y}{2} = \frac{84.8 \text{ MPa} - 28.9 \text{ MPa}}{2} = 27.9 \text{ MPa},$$

$$B = \frac{\sigma_x - \sigma_y}{2} = \frac{84.8 \text{ MPa} + 28.9 \text{ MPa}}{2} = 56.8 \text{ MPa}.$$

Now we substitute the first value of  $2\theta_p$  into Eq. (3.8) and obtain

$$\begin{aligned} \sigma_{x_1} &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta = \\ &= 27.9 \text{ MPa} + (56.8 \text{ MPa})(\cos 150.3^\circ) - (32.4 \text{ MPa})(\sin 150.3^\circ) = -37.5 \text{ MPa}. \end{aligned}$$

By the similar way, we substitute the second value of  $2\theta_p$  and obtain  $\sigma_{x_1} = 93.4 \text{ MPa}$ . In result, the principal stresses and their corresponding principal angles are

$$\begin{aligned} \sigma_1 &= 93.4 \text{ MPa} \text{ and } \theta_{p_1} = 165.2^\circ \\ \sigma_3 &= -37.5 \text{ MPa} \text{ and } \theta_{p_2} = 75.2^\circ. \end{aligned}$$

Keep in mind, that  $\sigma_2 = 0$  acts in  $z$  direction.

**Note that  $\theta_{p1}$  and  $\theta_{p2}$  differ by  $90^\circ$  and that  $\sigma_1 + \sigma_3 = \sigma_x + \sigma_y$ .**

The principal stresses are shown on a properly oriented element in the Fig. b. Of course, the principal planes are free from shear stresses.

The principal stresses may also be calculated directly from Eq. (3.26):

$$\begin{aligned}\sigma_{1,2(3)} &= \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \\ &= 27.9 \text{ MPa} \pm \sqrt{(56.8 \text{ MPa})^2 + (-32.4 \text{ MPa})^2}, \\ \sigma_{1,2(3)} &= 27.9 \text{ MPa} \pm 65.4 \text{ MPa}.\end{aligned}$$

Therefore,

$$\sigma_1 = 93.4 \text{ MPa}, \quad \sigma_3 = -37.5 \text{ MPa}, \quad (\sigma_2 = 0).$$

(2) *Maximum shear stresses.* The maximum in-plane shear stresses are given by Eq. (3.38):

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{(56.8 \text{ MPa})^2 + (-32.4 \text{ MPa})^2} = 65.4 \text{ MPa}.$$

The angle  $\theta_{s_1}$  to the plane having the maximum positive shear stress is calculated from Eq. (3.37):

$$\theta_{s_1} = \theta_{p_1} - 45^\circ = 165.2^\circ - 45^\circ = 120.2^\circ.$$

It follows that the maximum negative shear stress acts on the plane for which  $\theta_{s_2} = 120.2^\circ - 90^\circ = 30.2^\circ$ .

The normal stresses acting on the planes of maximum shear stresses are calculated from Eq. (3.40):

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} = 27.9 \text{ MPa}.$$

Finally, the maximum shear stresses and associated normal stresses are shown on the stress element of Fig. c.

### Example 3.6

The plane stress state is described in the figure. (1) Write the principal-stress-transformation formulas. (2) Calculate the principal stresses. (3) Calculate the maximum shearing stresses and the associated normal stresses. Sketch the results on properly oriented elements.

**Solution** (1) *Principal-stress-transformation formulas.*

Equations (3.26), (3.30), and (3.8) are written as

$$\sigma_1 = A + (B^2 + C^2)^{1/2},$$

$$\sigma_3 = A - (B^2 + C^2)^{1/2},$$

$$\theta_p = \frac{1}{2} \arctan \frac{C}{B},$$

$$\sigma_{x_1} = A + B \cos 2\theta + C \sin 2\theta,$$

where

$$A = \frac{1}{2}(\sigma_x + \sigma_y), \quad B = \frac{1}{2}(\sigma_x - \sigma_y), \quad C = \tau_{xy}.$$

(2) *Calculation of principal stresses.*

The principal angles  $\theta_p$  that locate the principal planes can be obtained from Eq. (3.19):

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(+5 \text{ MPa})}{7 \text{ MPa} - (+2 \text{ MPa})} = +2.0.$$

Solving for the angles, we get the following two sets of values:

$$2\theta_p = 63.4^\circ \text{ and } \theta_p = 31.7^\circ,$$

$$2\theta_p = 243.4^\circ \text{ and } \theta_p = 121.7^\circ.$$

The principal stresses may be obtained by substituting the two values of  $2\theta_p$  into the transformation equation for  $\sigma_{x_1}$  (Eq. 3.8). Determine preliminary the following quantities:

$$A = \frac{\sigma_x + \sigma_y}{2} = \frac{7 \text{ MPa} + 2 \text{ MPa}}{2} = 4.5 \text{ MPa},$$

$$B = \frac{\sigma_x - \sigma_y}{2} = \frac{7 \text{ MPa} - 2 \text{ MPa}}{2} = 2.5 \text{ MPa}.$$

Now we substitute the first value of  $2\theta_p$  into Eq. (3.8) and obtain

$$\begin{aligned} \sigma_{x_1} &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta = \\ &= A + B \cos 2\theta + C \sin 2\theta = \\ &= 4.5 \text{ MPa} + (2.5 \text{ MPa})(\cos 63.4^\circ) + (5 \text{ MPa})(\sin 63.4^\circ) = +10.09 \text{ MPa}. \end{aligned}$$

By the similar way, we substitute the second value of  $2\theta_p$  and obtain  $\sigma_{x_1} = -1.09$  MPa. In result, the principal stresses and their corresponding principal angles are

$$\begin{aligned}\sigma_1 &= +10.09 \text{ MPa and } \theta_{p_1} = 31.7^\circ \\ \sigma_3 &= -1.09 \text{ MPa and } \theta_{p_2} = 121.7^\circ.\end{aligned}$$

**Note that  $\theta_{p_1}$  and  $\theta_{p_2}$  differ by  $90^\circ$  and that  $\sigma_1 + \sigma_3 = \sigma_x + \sigma_y$ .**

The principal stresses are shown on a properly oriented element in Fig. b. Of course, the principal planes are free from shear stresses.

The principal stresses may also be calculated directly from the Eq. (3.26):

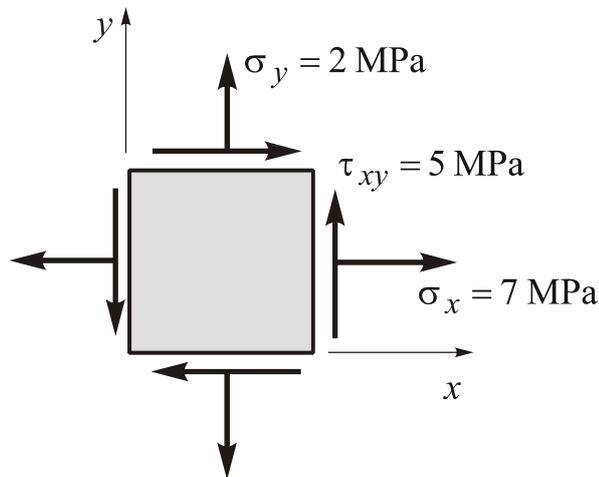
$$\begin{aligned}\sigma_{1,2(3)} &= \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = A \pm \sqrt{B^2 + C^2} = \\ &= 4.5 \text{ MPa} \pm \sqrt{(2.5 \text{ MPa})^2 + (5.0 \text{ MPa})^2}. \\ \sigma_{1,2(3)} &= 4.5 \text{ MPa} \pm 5.59 \text{ MPa}.\end{aligned}$$

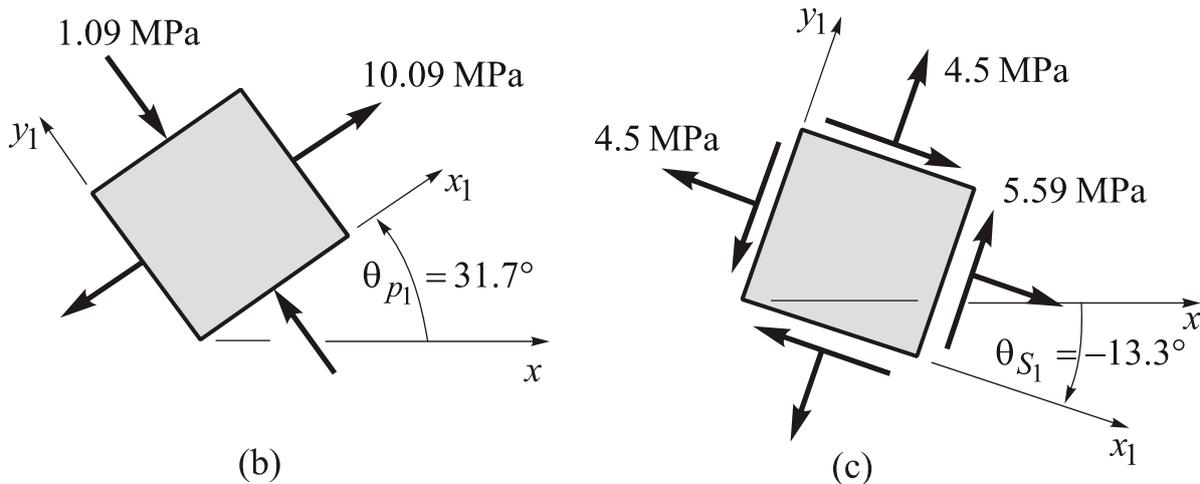
Therefore,

$$\sigma_1 = 10.09 \text{ MPa}, \quad \sigma_3 = -1.09 \text{ MPa} \quad (\sigma_2 = 0).$$

(3) *Maximum shear stresses.* The maximum in-plane shear stresses are given by Eq. (3.38):

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{7-2}{2}\right)^2 + 5^2} = 5.59 \text{ MPa}.$$





The angle  $\theta_{s_1}$  to the plane having the maximum positive shear stress is calculated from Eq. (3.37):

$$\theta_{s_1} = \theta_{p_1} - 45^\circ = 31.7^\circ - 45^\circ = -13.3^\circ.$$

It follows that the maximum negative shear stress acts on the plane for which  $\theta_{s_2} = -13.3^\circ + 90^\circ = 76.7^\circ$ .

The normal stresses acting on the planes of maximum shear stresses are calculated from Eq. (3.40):

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} = 4.5 \text{ MPa}.$$

Finally, the maximum shear stresses and associated normal stresses are shown on the stress element of Fig. c.

**Note.** The direction of the  $\tau_{\text{max}}$  may also be readily predicted by recalling that they act toward the shear diagonal.



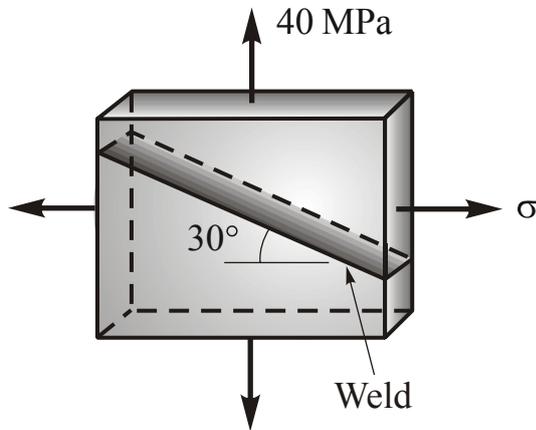
## PROBLEMS

**Problem 3.34** Calculate the stresses on planes of maximum shearing stresses for an element subjected to principal stresses: (1)  $\sigma = 60 \text{ MPa}$  and  $\sigma = 20 \text{ MPa}$ ; (2)  $\sigma = 30 \text{ MPa}$  and  $\sigma = -7 \text{ MPa}$ ; (3)  $\sigma = -10 \text{ MPa}$ , and  $\sigma = -30 \text{ MPa}$ .

**Problem 3.35** A cylindrical tank fabricated of 10-mm-thick plate is subjected to an internal pressure of 6 MPa. (1) Determine the maximum diameter if the maximum shear stress is limited to 30 MPa. (2) For the diameter found in part (1) determine the limiting value of tensile stress. *In the solution*

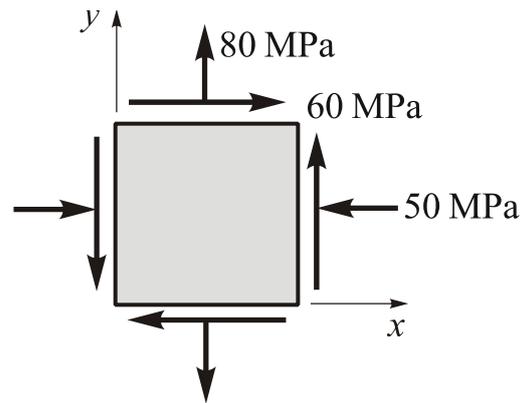
use that circumferential stress is equal to  $\sigma_c = pr/t$  ( $p$  – pressure,  $r$  – radius,  $t$  – thickness), and axial stress is equal to  $\sigma_a = pr/2t$ .

**Problem 3.36** A welded plate is subjected to the uniform biaxial tension shown in the figure. Calculate the maximum stress  $\sigma$ , if the weld has (1) an allowable shear stress of 25 MPa and (2) an allowable normal stress of 60 MPa.

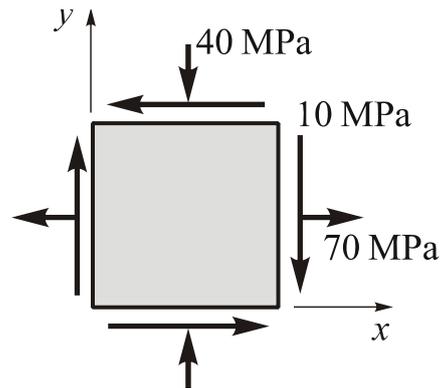


**Problem 3.37** A closed cylindrical vessel, constructed of a thin plate 1.5 m in diameter, is subjected to an external pressure of 1 MPa. Calculate (1) the wall thickness if the maximum allowable shear stress is set at 20 MPa and (2) the corresponding maximum principal stress. *In the solution use that circumferential stress is equal to  $\sigma_c = pr/t$  ( $p$  – pressure,  $r$  – radius,  $t$  – thickness), and axial stress is equal to  $\sigma_a = pr/2t$ .*

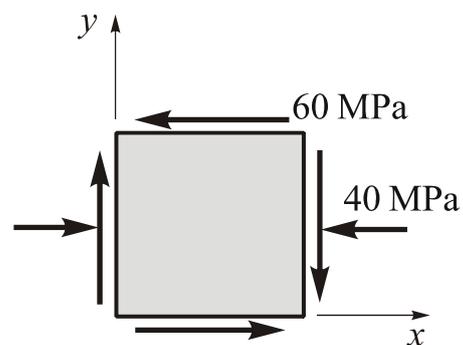
**Problem 3.38** For the state of stress given in the figure, determine the magnitude and orientation of the principal stresses. Show the results on a properly oriented element.



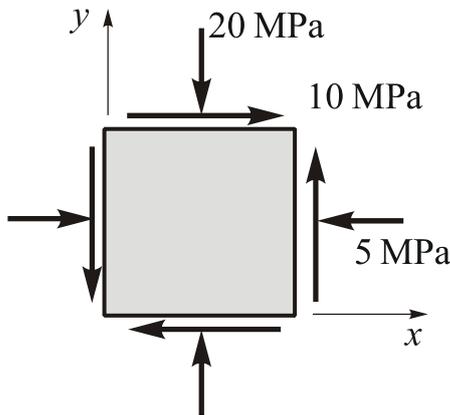
**Problem 3.39** Determine the maximum shearing stresses and the associated normal stresses for the state of stress represented in the figure. Sketch the results on a properly oriented element.



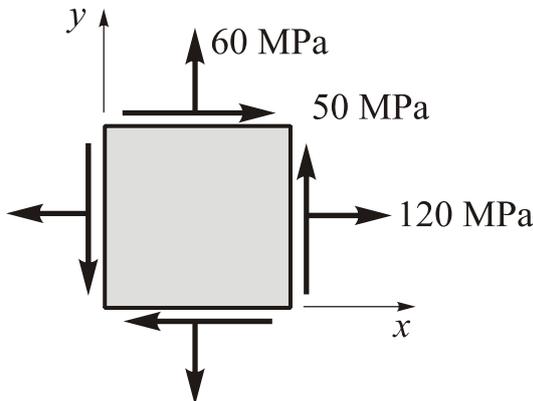
**Problem 3.40** The stresses at the point in a loaded member are represented in the figure. Calculate and sketch (1) the principal stresses and (2) the maximum shearing stresses with the associated normal stresses.



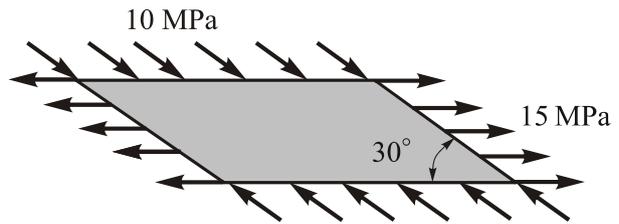
**Problem 3.41** The stresses at the point in a loaded member are represented in the figure. Calculate and sketch (1) the principal stresses and (2) the maximum shearing stresses with the associated normal stresses.



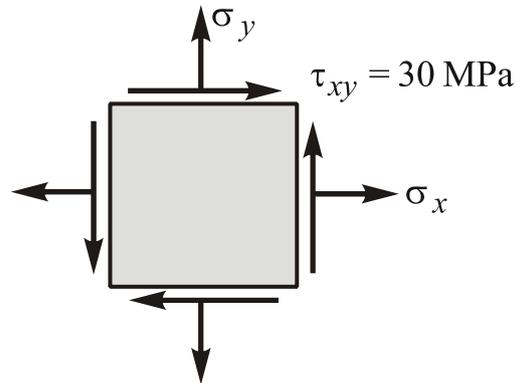
**Problem 3.42** The stresses at the point in a loaded member are represented in the figure. Calculate and sketch (1) the principal stresses and (2) the maximum shearing stresses with the associated normal stresses.



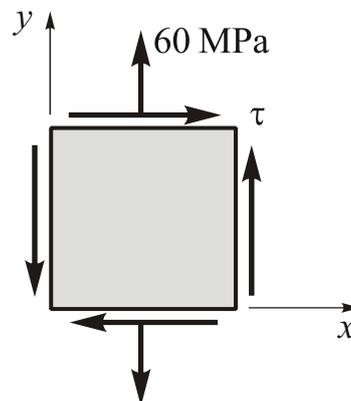
**Problem 3.43** Given the stresses acting uniformly at the edges of a block (see figure), calculate (1) the stresses  $\sigma_x, \sigma_y, \tau_{xy}$  and (2) the maximum shearing stresses with the associated normal stresses. Sketch the results on properly oriented elements.



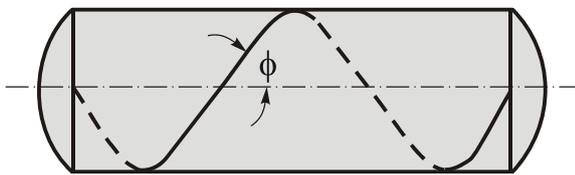
**Problem 3.44** The shearing stress at a point of the element's surface is  $\tau_{xy} = 30$  MPa (see figure). The principal stresses at this point are  $\sigma_1 = 35$  MPa and  $\sigma_3 = -55$  MPa. Determine the  $\sigma_x$  and  $\sigma_y$  and indicate the principal and maximum shear stresses on an appropriate sketch.



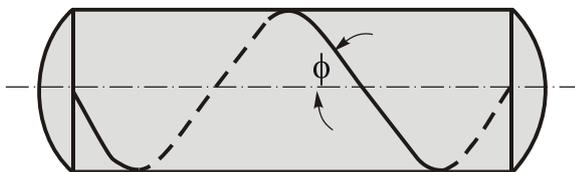
**Problem 3.45** The state of stress at a point is shown in the figure. Determine (1) the magnitude of the shear stress  $\tau$  if the maximum principal stress is not to exceed 70 MPa and (2) the corresponding maximum shearing stresses and the planes at which they act.



**Problem 3.46** The side wall of the cylindrical steel pressure vessel has butt-welded seams (see figure). The allowable tensile strength of the joint is 80 % of that of steel. Determine the maximum value of the seam angle  $\phi$  if the tension in the steel is to be limiting. In the solution use that circumferential stress is equal to  $\sigma_c = pr/t$  ( $p$  – pressure,  $r$  – radius,  $t$  – thickness), and axial stress is equal to  $\sigma_a = pr/2t$ .

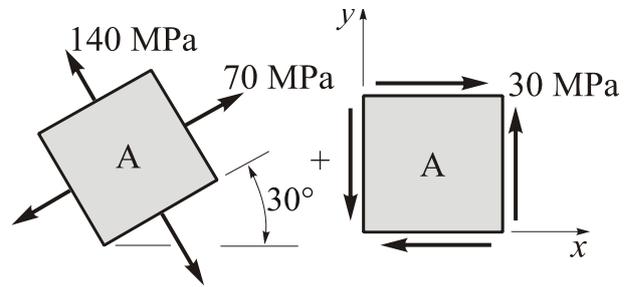


**Problem 3.47** A cylindrical vessel of internal diameter 300 mm and wall thickness 3 mm has a welded helical seam angle of  $\phi = 60^\circ$  (see figure). If the allowable tensile stress in the weld is 100 MPa, determine (1) the maximum value of internal pressure  $p$  and (2) the corresponding shear stress in the weld. In the solution use that circumferential stress is equal to  $\sigma_c = pr/t$  ( $p$  – pressure,  $r$  – radius,  $t$  – thickness), and axial stress is equal to  $\sigma_a = pr/2t$ .

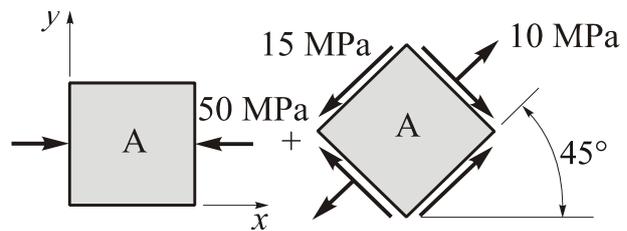


**Problem 3.48** A structural member is subjected to two different loadings, each separately producing stresses at point  $A$ , as indicated in the figure. Calculate, and show on a sketch, the principal planes and the principal

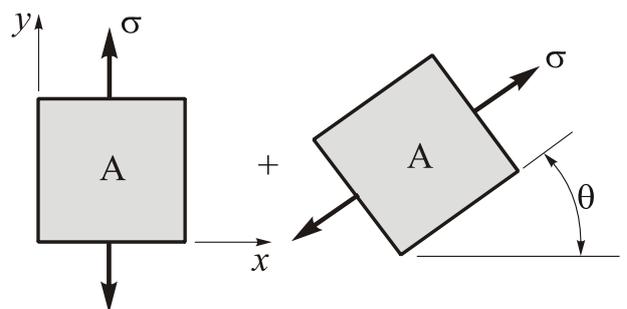
stresses under the effect of the combined loading.



**Problem 3.49** A structural member is subjected to two different loadings, each separately producing stresses at point  $A$ , as indicated in the figure. Calculate, and show on a sketch, the principal planes and the principal stresses under the effect of the combined loading.

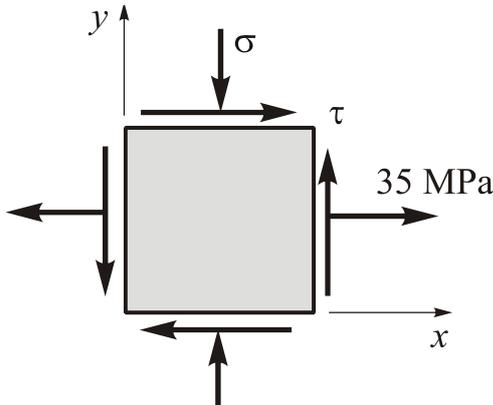
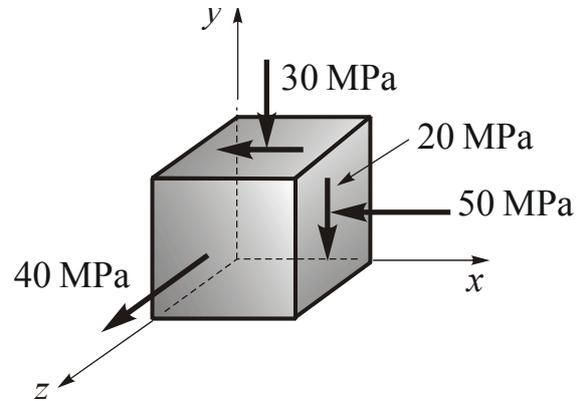


**Problem 3.50** A structural member is subjected to two different loadings, each separately producing stresses at point  $A$ , as indicated in the figure. Calculate the principal planes and the principal stresses under the effect of the combined loading, if  $\sigma$  and  $\theta$  are known constants.



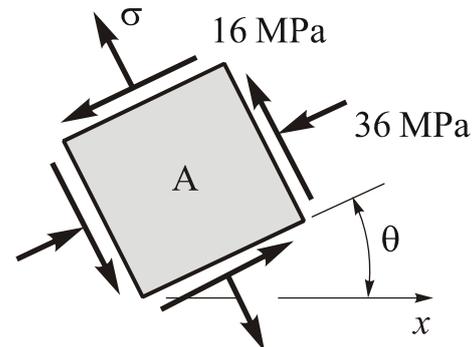
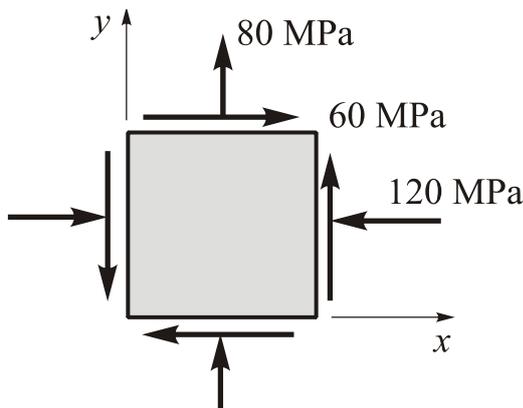
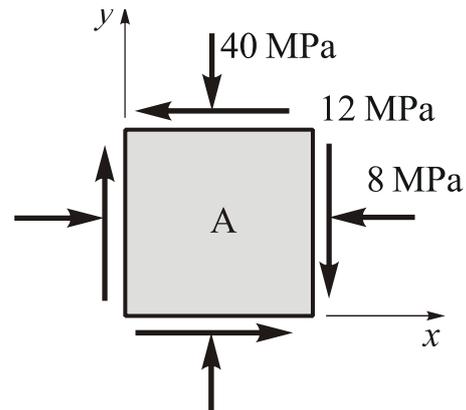
**Problem 3.51** The state of stress on the horizontal and vertical planes at a point is only incompletely known, as shown in the figure. However, at the point  $\sigma_3$  and  $\tau_{\max}$  are prescribed as  $-50$  and  $55$  MPa, respectively. Determine stresses  $\sigma$ ,  $\tau$  and  $\sigma_1$ . Show the results on properly oriented elements.

Sketch the results on properly oriented elements.



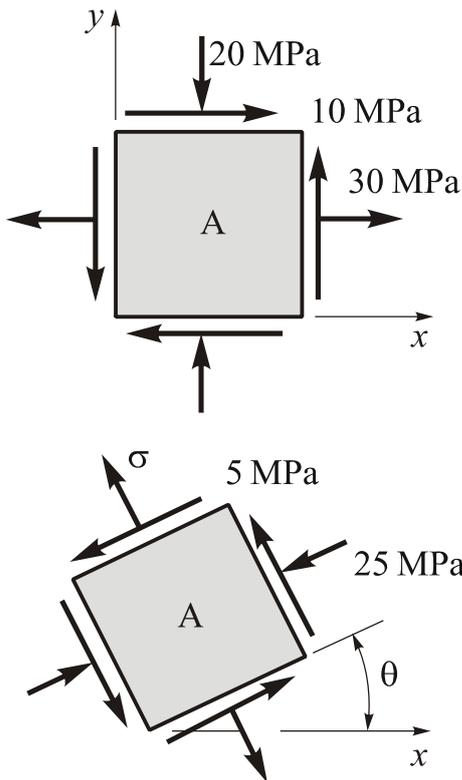
**Problem 3.54** The state of stress at a point *A* in a structure is shown in the figure. Determine the normal stress  $\sigma$  and the angle  $\theta$ .

**Problem 3.52** The stresses at the point in a loaded member are represented in the figure. Calculate and sketch (1) the principal stresses and (2) the maximum shearing stresses with the associated normal stresses.

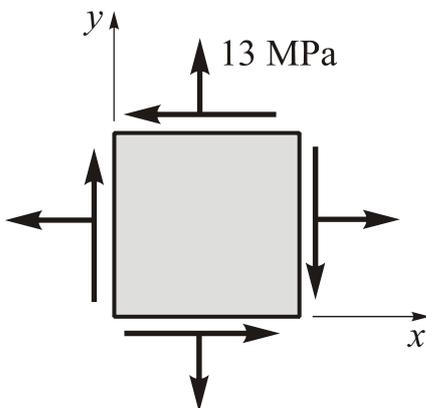


**Problem 3.53** Consider a point in a loaded solid subjected to the stresses shown in the figure. Determine (1) the principal stresses and (2) the

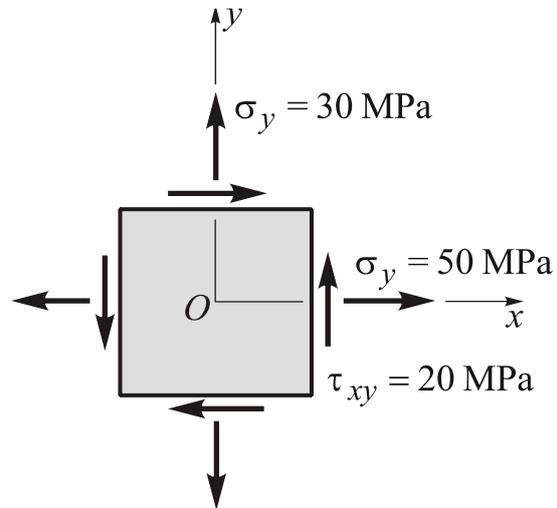
**Problem 3.55** The state of stress at a point *A* in a structure is shown in the figure. Determine the normal stress  $\sigma$  and the angle  $\theta$ .



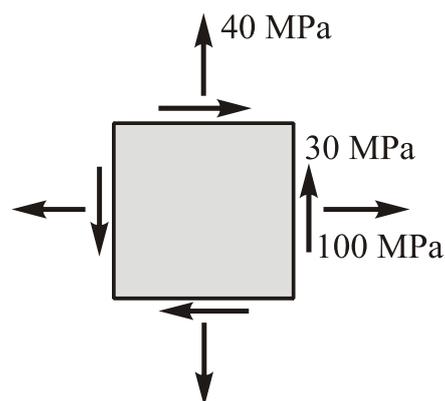
**Problem 3.56** At a point of a loaded member (see figure) a normal stress  $\sigma_y = 13$  MPa and a negative shear stress on the horizontal plane exist. One of the principal stresses at the point is 7 MPa (tension), and the maximum shearing stress has a magnitude of 35 MPa. Calculate (1) the unknown stresses on the horizontal and vertical planes and (2) the unknown principal stress. Show the principal stresses and maximum shear stresses on a sketch of a properly oriented element.



**Problem 3.57** An element in plane stress is subjected to stresses  $\sigma_x = 50$  MPa,  $\sigma_y = 30$  MPa, and  $\tau_{xy} = 20$  MPa (see figure). Determine the principal stresses and show them on a sketch of a properly oriented element.

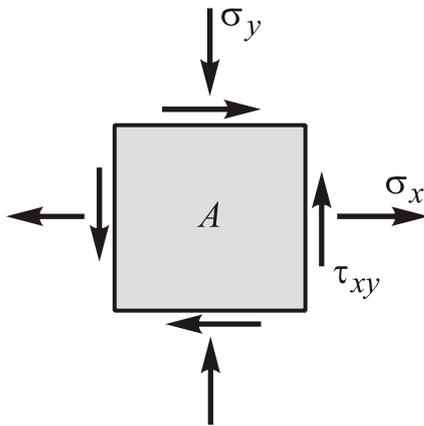
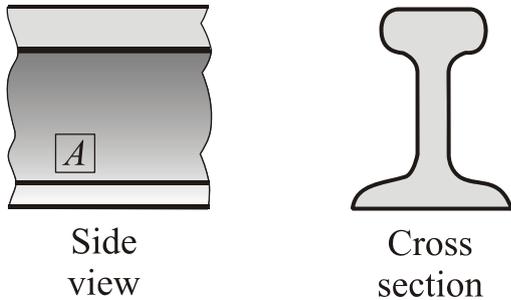


**Problem 3.58** An element in plane stress is subjected to stresses  $\sigma_x = 100$  MPa,  $\sigma_y = 40$  MPa, and  $\tau_{xy} = 30$  MPa (see figure). Determine the principal stresses and show them on a sketch of a properly oriented element.

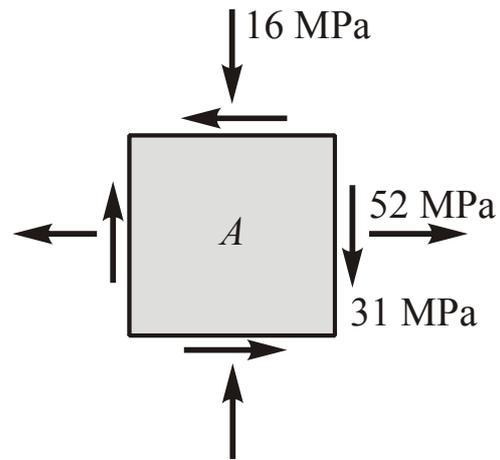
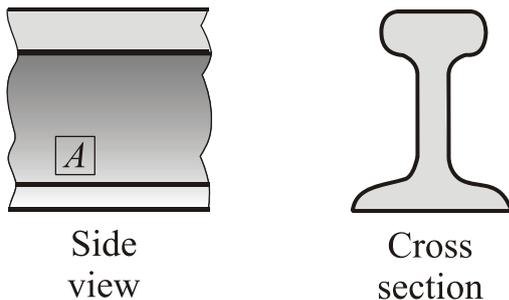


**Problem 3.59** An element in plane stress is subjected to stresses  $\sigma_x = 65$  MPa,  $\sigma_y = -150$  MPa, and  $\tau_{xy} = 35$  MPa (see figure). Determine

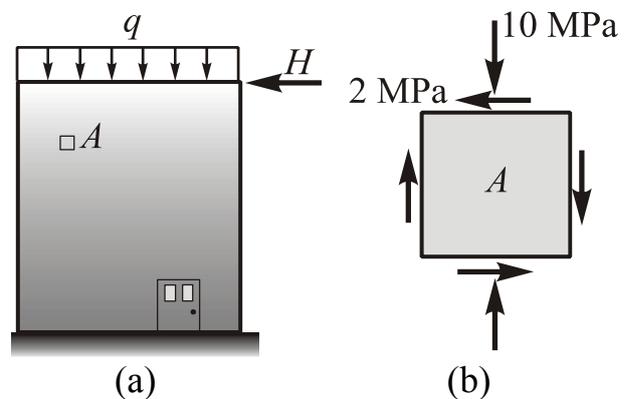
the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.



**Problem 3.60** An element in plane stress is subjected to stresses  $\sigma_x = 52 \text{ MPa}$ ,  $\sigma_y = -16 \text{ MPa}$ , and  $\tau_{xy} = -31 \text{ MPa}$  (see figure). Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

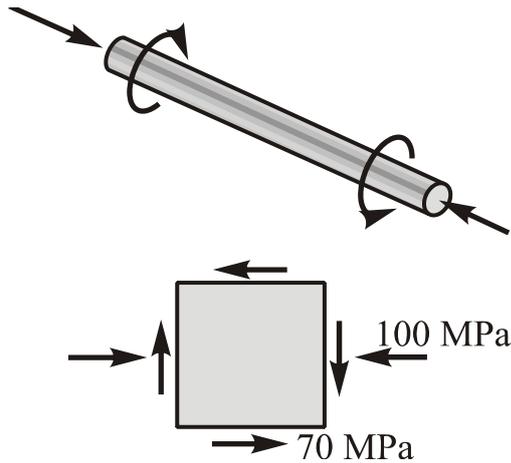


**Problem 3.61** A shear wall in a reinforced concrete building is subjected to a vertical uniform load  $q$  and a horizontal force  $H$ , as shown in (a) part of the figure. The stresses at point  $A$  on the surface of the wall have the values shown in (b) part of the figure (compressive stress equal to 10 MPa and shear stress equal to 2 MPa). (1) Determine the principal stresses and show them on a sketch of a properly oriented element. (2) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

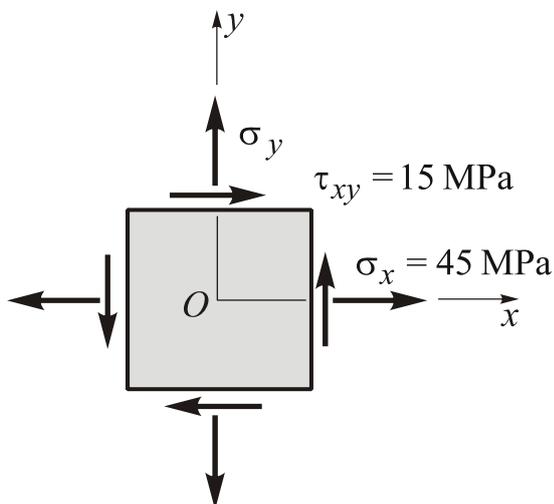


**Problem 3.62** A engine shaft subjected to combined torsion and axial thrust is designed to resist a shear stress

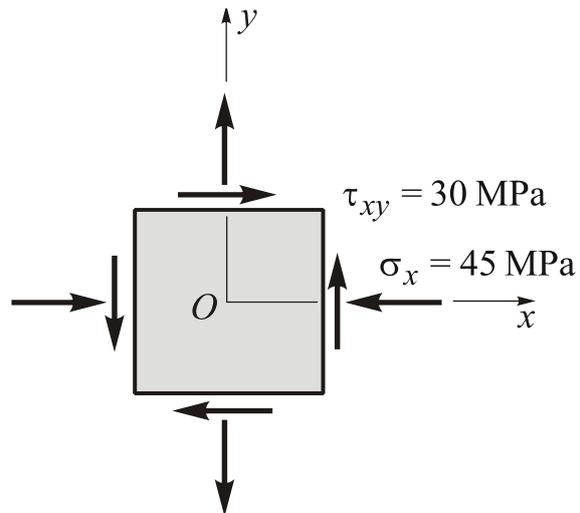
of 70 MPa and a normal stress of 100 MPa (see figure). (1) Determine the principal stresses and show them on a sketch of a properly oriented element. (2) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.



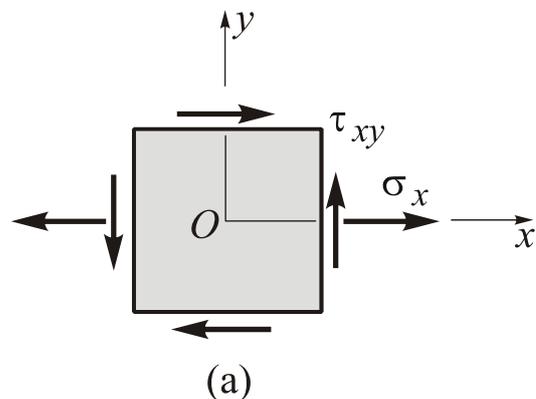
**Problem 3.63** At a point on the surface of a machine component the stresses acting on the  $x$  face of a stress element are  $\sigma_x = 45$  MPa and  $\tau_{xy} = 15$  MPa (see figure). What is the allowable range of values for the stress  $\sigma_y$  if the maximum shear stress is limited to  $\tau_{\text{lim}} = 20$  MPa?

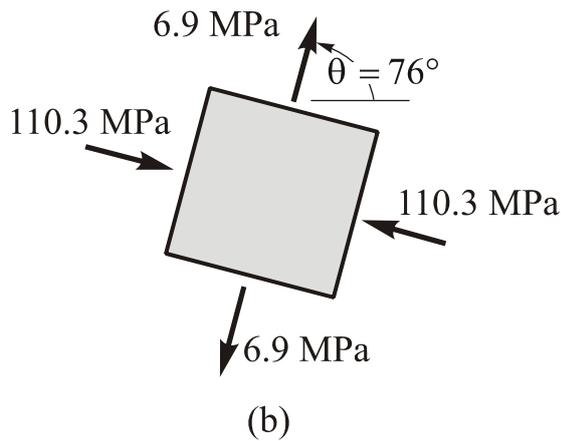


**Problem 3.64** At a point on the surface of a structure the stresses acting on the  $x$  face of a stress element are  $\sigma_x = -45$  MPa and  $\tau_{xy} = 30$  MPa (see figure). What is the allowable range of values for the stress  $\sigma_y$  if the maximum shear stress is limited to  $\tau_{\text{lim}} = 35$  MPa?

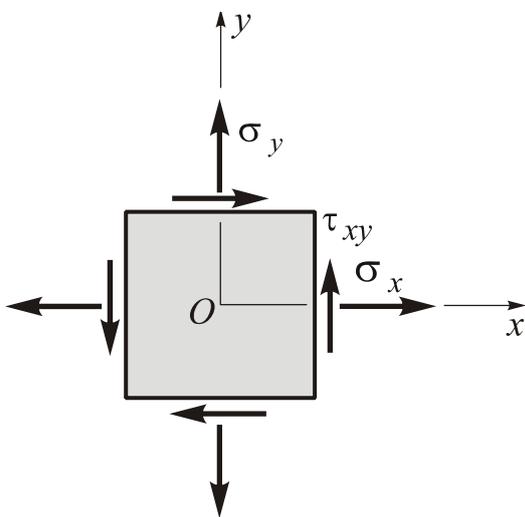


**Problem 3.65** At a point in the web of a steel beam the stresses consist of a normal stress  $\sigma_x$  and a shear stress  $\tau_{xy}$ , as shown in part (a) of the figure. At this same point, the principal stresses are 6.9 MPa (tension) at an angle  $\theta = 76.0^\circ$  and 110.3 MPa (compression) (see part (b) of the figure). Determine the stresses  $\sigma_x$  and  $\tau_{xy}$  acting on the  $xy$  element.



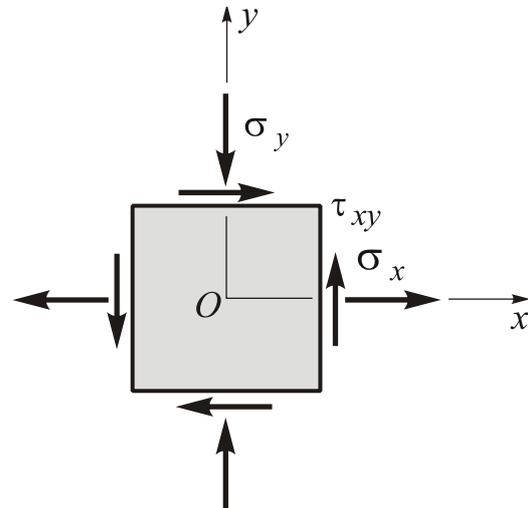


**Problem 3.66** An element in plane stress (see figure) is subjected to stresses  $\sigma_x = 72$  MPa,  $\sigma_y = 30$  MPa, and  $\tau_{xy} = 28$  MPa. (1) Determine the principal stresses and show them on a sketch of a properly oriented element. (2) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

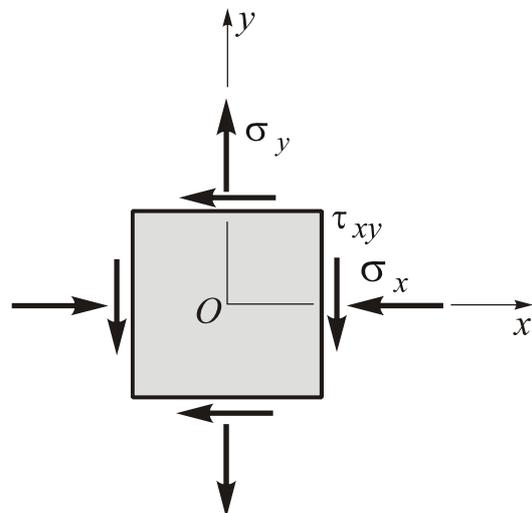


**Problem 3.67** An element in plane stress (see figure) is subjected to stresses  $\sigma_x = 45$  MPa,  $\sigma_y = -185$  MPa, and  $\tau_{xy} = 55$  MPa. (1) Determine the principal stresses and

show them on a sketch of a properly oriented element. (2) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.



**Problem 3.68** An element in plane stress (see figure) is subjected to stresses  $\sigma_x = -80$  MPa,  $\sigma_y = 130$  MPa, and  $\tau_{xy} = -35$  MPa. (1) Determine the principal stresses and show them on a sketch of a properly oriented element. (2) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

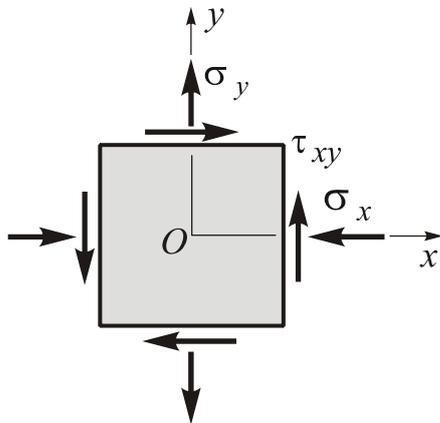


**Problem 3.69** An element in plane stress (see figure) is subjected to stresses

$$\sigma_x = -75 \text{ MPa}, \quad \tau_{xy} = 200 \text{ MPa},$$

$$\sigma_y = 215 \text{ MPa},$$

(1) Determine the principal stresses and show them on a sketch of a properly oriented element. (2) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

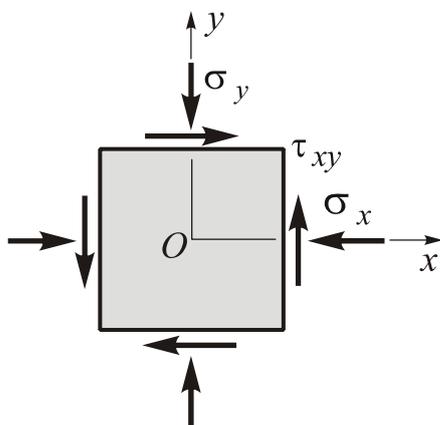


**Problem 3.70** An element in plane stress (see figure) is subjected to stresses

$$\sigma_x = -70 \text{ MPa}, \quad \tau_{xy} = 30 \text{ MPa},$$

$$\sigma_y = -35 \text{ MPa},$$

(1) Determine the principal stresses and show them on a sketch of a properly oriented element. (2) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

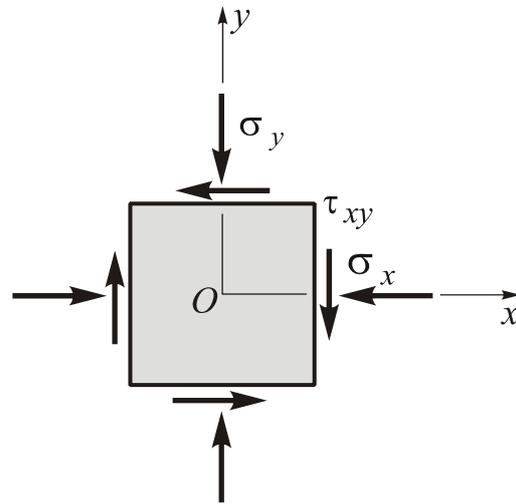


**Problem 3.71** An element in plane stress (see figure) is subjected to stresses

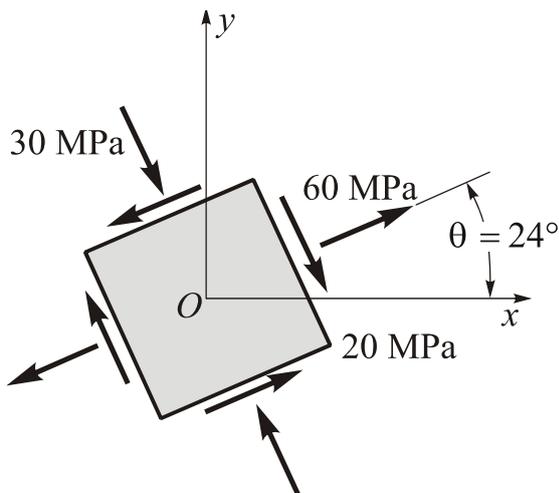
$$\sigma_x = -145 \text{ MPa}, \quad \tau_{xy} = -30 \text{ MPa},$$

$$\sigma_y = -30 \text{ MPa},$$

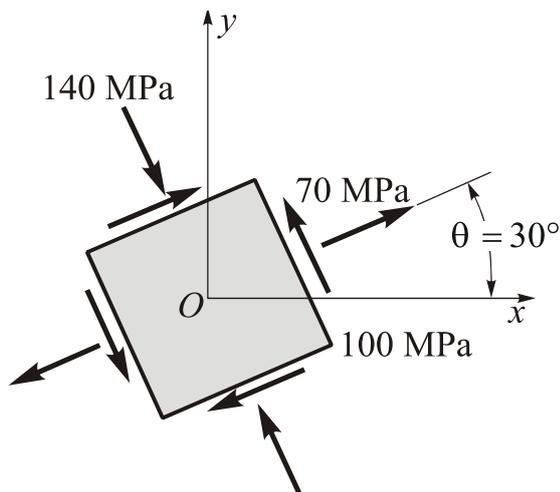
(1) Determine the principal stresses and show them on a sketch of a properly oriented element. (2) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.



**Problem 3.72** At a point on the surface of a machine, the stresses acting on an element oriented at an angle  $\theta = 24^\circ$  to the  $x$  axis have the magnitudes and directions shown in the figure. (1) Determine the principal stresses and show them on a sketch of a properly oriented element. (2) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

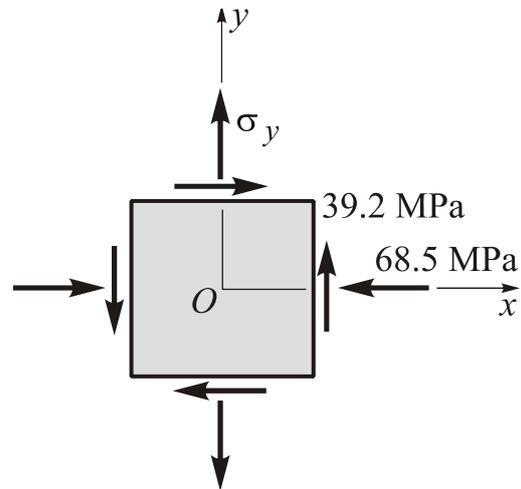


**Problem 3.73** At a point on the surface of a machine, the stresses acting on an element oriented at an angle  $\theta = 30^\circ$  to the  $x$  axis have the magnitudes and directions shown in the figure. (1) Determine the principal stresses and show them on a sketch of a properly oriented element. (2) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

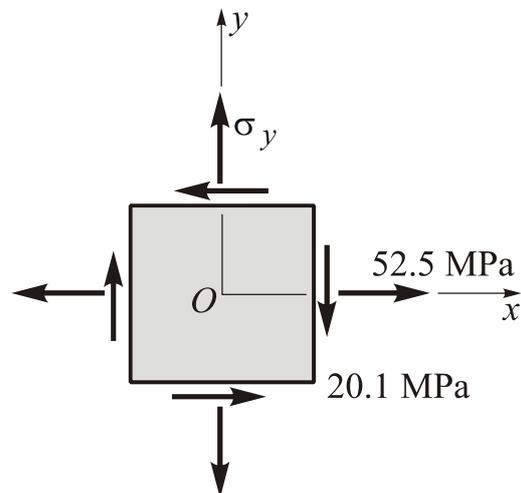


**Problem 3.74** An element in plane stress is subjected to stresses  $\sigma_x = -68.5 \text{ MPa}$  and  $\tau_{xy} = 39.2 \text{ MPa}$  (see the figure). It is known that one of the principal stresses equals  $26.1 \text{ MPa}$

in tension. (1) Determine the stress  $\sigma_y$ . (2) Determine the other principal stress and the orientation of the principal planes; then show the principal stresses on a sketch of a properly oriented element.



**Problem 3.75** An element in plane stress is subjected to stresses  $\sigma_x = 52.5 \text{ MPa}$  and  $\tau_{xy} = -20.1 \text{ MPa}$  (see figure). It is known that one of the principal stresses equals  $57.9 \text{ MPa}$  in tension. (1) Determine the stress  $\sigma_y$ . (2) Determine the other principal stress and the orientation of the principal planes; then show the principal stresses on a sketch of a properly oriented element.

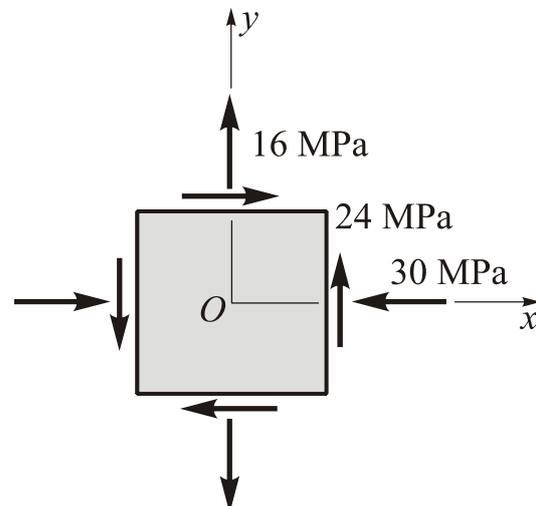
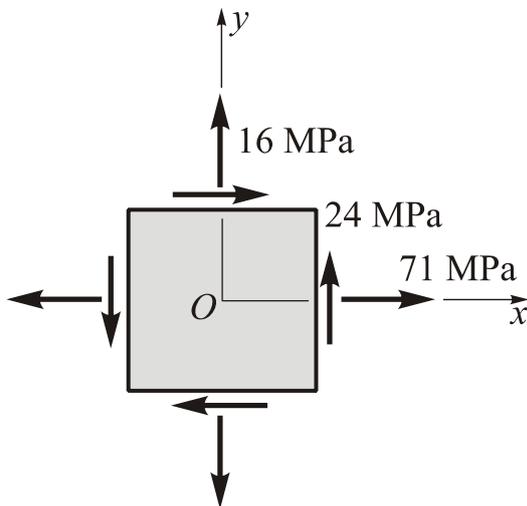


**Problem 3.76** An element in plane stress is subjected to stresses  $\sigma_x = 71$  MPa,  $\sigma_y = 16$  MPa, and  $\tau_{xy} = 24$  MPa (see figure).

(1) Determine the principal stresses and show them on a sketch of a properly oriented element. (2) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

**Problem 3.77** An element in plane stress is subjected to stresses  $\sigma_x = -30$  MPa,  $\sigma_y = 16$  MPa, and  $\tau_{xy} = 24$  MPa (see figure).

Determine the principal stresses and show them on a sketch of a properly oriented element. (2) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.



### 3.4 Mohr's Circle for Plane Stress

The basic equations of stress transformation derived earlier may be interpreted graphically. The graphical technique permits the rapid transformation of stress from one plane to another and also provides an overview of the state of stress at a point. It provides a means for calculating principal stresses, maximum shear stresses, and stresses on inclined planes. This method was devised by the German civil engineer Otto Christian Mohr (1835–1918), who developed a plot known as *Mohr's circle* in 1882. Mohr's circle is valid not only for stresses, but also for other quantities of a similar nature, including strains and moments of inertia.

The equations of Mohr's circle can be derived from the transformation equations for plane stress (Eqs. (3.8), (3.9)). These two equations may be represented as

$$\sigma_{x_1} - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta, \quad (3.41)$$

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta. \quad (3.42)$$

Squaring each equation, adding them, and simplifying, we obtain well-known *equation of a circle*:

$$\left( \sigma_{x_1} - \frac{\sigma_x + \sigma_y}{2} \right)^2 + \tau_{x_1y_1}^2 = \left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2. \quad (3.43)$$

This equation can be written in more simple form using the following notation:

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2}, \quad (3.44)$$

$$R = \sqrt{\left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2}. \quad (3.45)$$

Equation (3.43) now becomes

$$\left( \sigma_{x_1} - \sigma_{\text{aver}} \right)^2 + \tau_{x_1y_1}^2 = R^2, \quad (3.46)$$

which is the equation of a circle in standard algebraic form. The coordinates are  $\sigma_{x_1}$  and  $\tau_{x_1y_1}$ , the radius is  $R$  and the center of the circle has coordinates  $\sigma_{x_1} = \sigma_{\text{aver}}$  and  $\tau_{x_1y_1} = 0$ .

Mohr's circle can be plotted from Eqs. (3.41, 3.42) and (3.46) in two different ways. In our form of Mohr's circle we will plot the normal stress  $\sigma_{x_1}$  positive to the right and the shear stress  $\tau_{x_1y_1}$  positive downward, as shown in Fig. 3.18. *The advantage of plotting shear stresses positive downward is that the angle  $2\theta$  on Mohr's circle is positive when counterclockwise, which agrees with the positive direction of  $2\theta$  in the derivation of the transformation equations.*

Mohr's circle can be constructed in a variety of ways, depending upon which stresses are known and which are unknown. Let us assume that we know the stresses  $\sigma_x$ ,  $\sigma_y$ , and  $\tau_{xy}$  acting on the  $x$  and  $y$  planes of an element in plane stress (Fig. 3.19a). This information is sufficient to construct the circle.

Then, with the circle drawn, we can determine the stresses  $\sigma_{x_1}$ ,  $\sigma_{y_1}$ , and  $\tau_{x_1y_1}$  acting on an inclined element (Fig. 3.19b). We can also obtain the principal stresses and maximum shear stresses from the circle.

With  $\sigma_x$ ,  $\sigma_y$ , and  $\tau_{xy}$  known, the procedure for constructing Mohr's circle is as follows (see Fig. 3.19c):

(a) Draw a set of coordinate axes with  $\sigma_{x_1}$  as abscissa (positive to the right) and  $\tau_{x_1y_1}$  as ordinate (positive downward).

(b) Locate the center  $C$  of the circle at the point having coordinates  $\sigma_{x_1} = \sigma_{\text{aver}}$  and  $\tau_{x_1y_1} = 0$  (see Eqs. (3.44) and (3.46)).

(c) Locate point  $A$ , representing the stress conditions on the  $x$  face of the element shown in Fig. 3.19a, by plotting its coordinates  $\sigma_{x_1} = \sigma_x$  and  $\tau_{x_1y_1} = \tau_{xy}$ . Note that point  $A$  corresponds to  $\theta = 0$ . The  $x$  face of the element (Fig. 3.19a) is labeled "A" to show its correspondence with point  $A$  in the diagram.

(d) Locate point  $B$  representing the stress conditions on the  $y$  face of the element shown in Fig. 3.19a, by plotting its coordinates  $\sigma_{x_1} = \sigma_y$  and  $\tau_{x_1y_1} = -\tau_{xy}$ . Point  $B$  corresponds to  $\theta = 90^\circ$ . The  $y$  face of the element (Fig. 3.19a) is labeled "B" to show its correspondence with point  $B$  in the diagram.

(e) Draw a line from point  $A$  to point  $B$ . It is a diameter of the circle and passes through the center  $C$ . Points  $A$  and  $B$ , representing the stresses on planes at  $90^\circ$  to each other, are at opposite ends of the diameter (and therefore are  $180^\circ$  apart on the circle).

(f) Using point  $C$  as the center, draw Mohr's circle through points  $A$  and  $B$ . The circle drawn in this manner has radius  $R$  (Eq. (3.45)).

**Note.** When Mohr's circle is plotted to scale, numerical results can be obtained graphically.

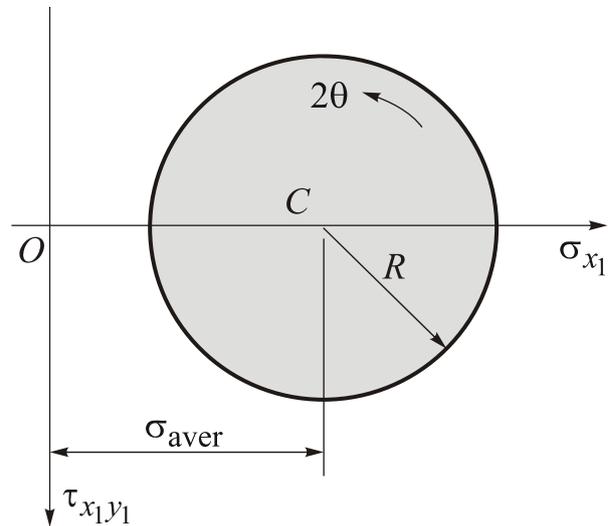


Fig 3.18 The form of Mohr's circle with  $\tau_{x_1y_1}$  positive downward and the angle  $2\theta$  positive counterclockwise

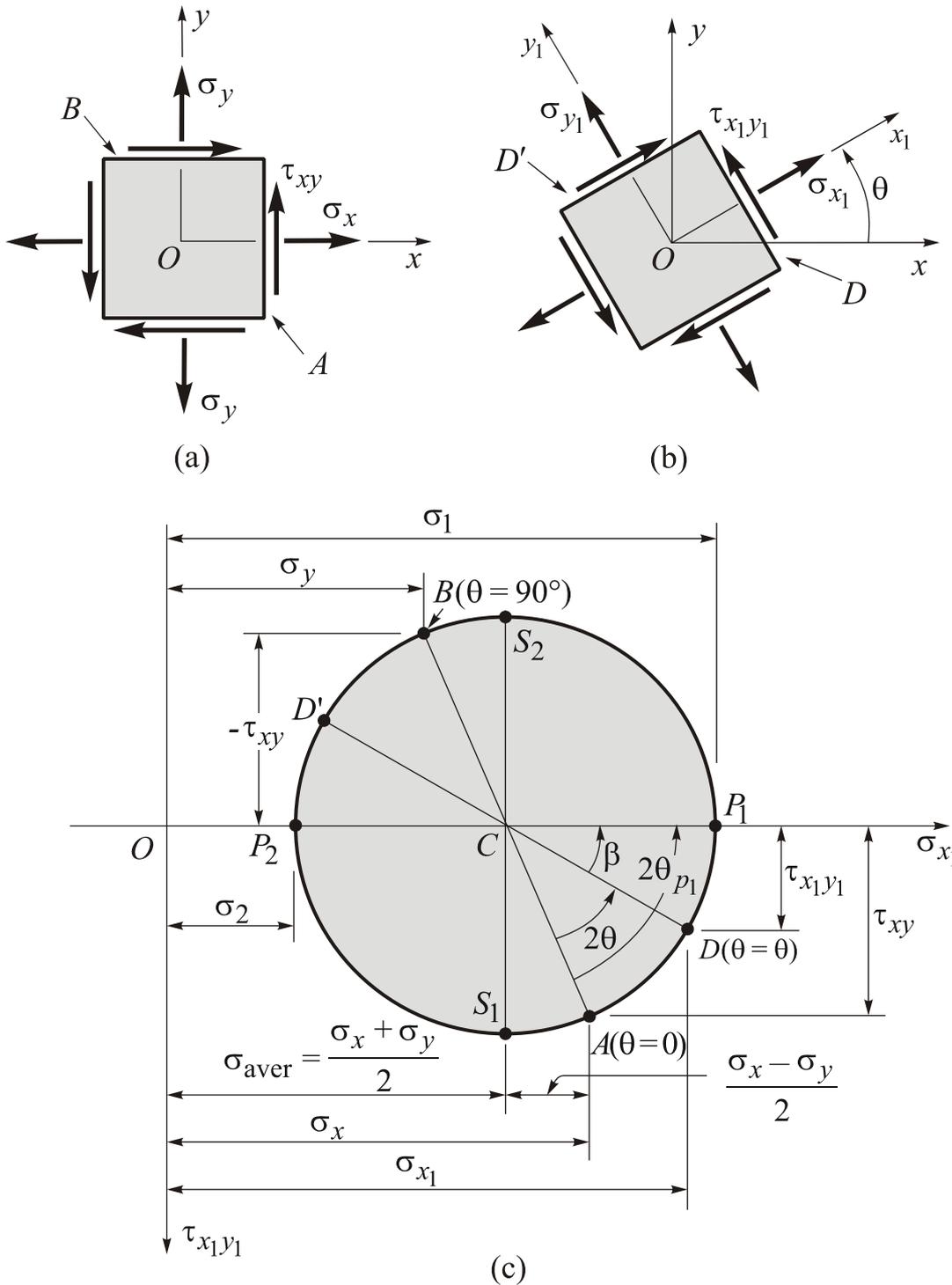


Fig. 3.19 Construction of Mohr's circle for plane stress

(1) *Stresses on an inclined element.* Mohr's circle shows how the stresses represented by points on it are related to the stresses acting on an element. The stresses on an inclined plane defined by the angle  $\theta$  (Fig. 3.19b) are found on the circle at the point where the angle from the reference point (point A) is  $2\theta$ . Thus,

as we rotate the  $x_1y_1$  axes counterclockwise through an angle  $\theta$  (Fig. 3.19b), the point on Mohr's circle corresponding to the  $x_1$  face moves counterclockwise through an angle  $2\theta$ . Similarly, in clockwise rotation of the axes, the point on the circle moves clockwise through an angle twice as large.

(2) *Principal stresses.* The determination of principal stresses is the most important application of Mohr's circle. As we move around Mohr's circle (Fig. 3.19c), we encounter point  $P_1$  where the normal stress reaches its algebraically largest value and the shear stress is zero. Hence, point  $P_1$  gives the algebraically larger principal stress and its angle  $2\theta_{p_1}$  from the reference point  $A$  ( $\theta = 0$ ) gives the orientation of the principal plane. The next principal plane, associated with the algebraically smallest normal stress, is represented by point  $P_2$ , diametrically opposite to point  $P_1$ .

(3) *Maximum shear stresses.* Points  $S_1$  and  $S_2$  which represent the planes of maximum positive and maximum negative shear stresses, respectively, are located at the bottom and top of Mohr's circle (Fig. 3.19c). These points are at angles  $2\theta = 90^\circ$  from points  $P_1$  and  $P_2$ , which agrees with the fact that the planes of maximum shear stress are oriented at  $45^\circ$  to the principal planes. The maximum shear stresses are numerically equal to the radius  $R$  of the circle. Also, the normal stresses on the planes of maximum shear stress are equal to the abscissa of point  $C$ , which is the average normal stress  $\sigma_{aver}$ .

Various multiaxial states of stress can readily be treated by applying the foregoing procedure. Fig. 3.20 shows some examples of Mohr's circles for commonly encountered cases. Analysis of material behavior subject to different loading conditions is often facilitated by this type of compilation. Interestingly, for the case of equal tension and compression (this type of stress state was named as pure shear) (see Fig. 3.20a),  $\sigma_z = 0$  and the  $z$ -directed strain does not exist ( $\varepsilon_z = 0$ ). Hence the element is in a state of *plane strain as well as plane stress*. An element in this condition can be converted to a condition of pure shear by rotating it  $45^\circ$  as indicated.

In the case of triaxial tension (Fig. 3.20b and 3.21a), a Mohr's circle is drawn corresponding to each projection of a three-dimensional element (see Fig. 3.21b). The three-circle cluster represents Mohr's circle for triaxial stress. The case of tension with lateral pressure (Fig. 3.20c) is explained similarly.

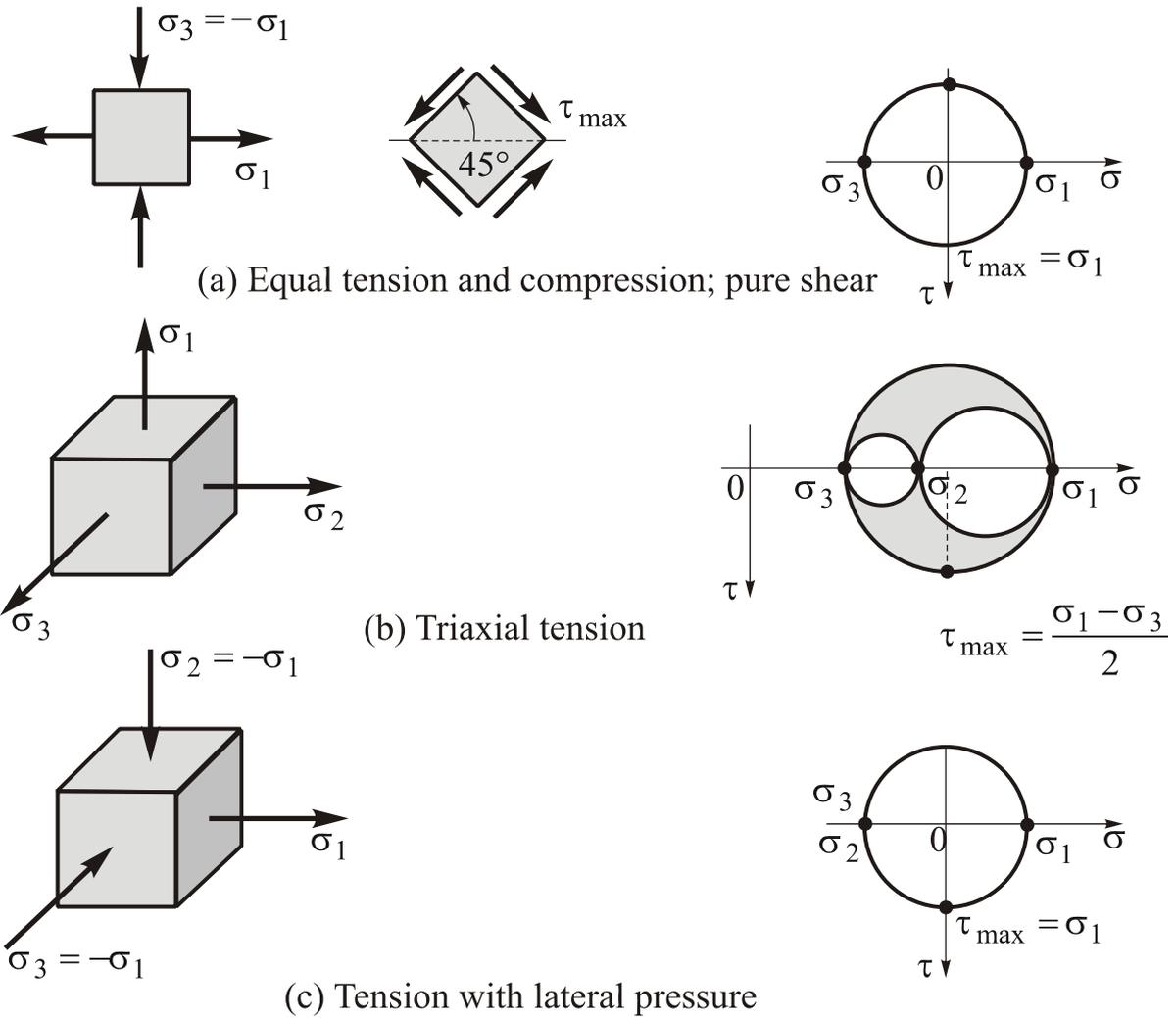


Fig. 3.20 Mohr's circle for various states of stress

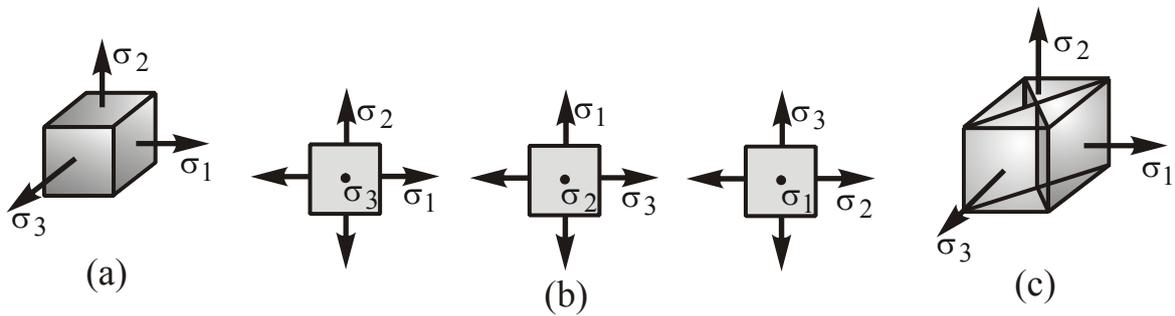


Fig. 3.21 Three-dimensional state of stress

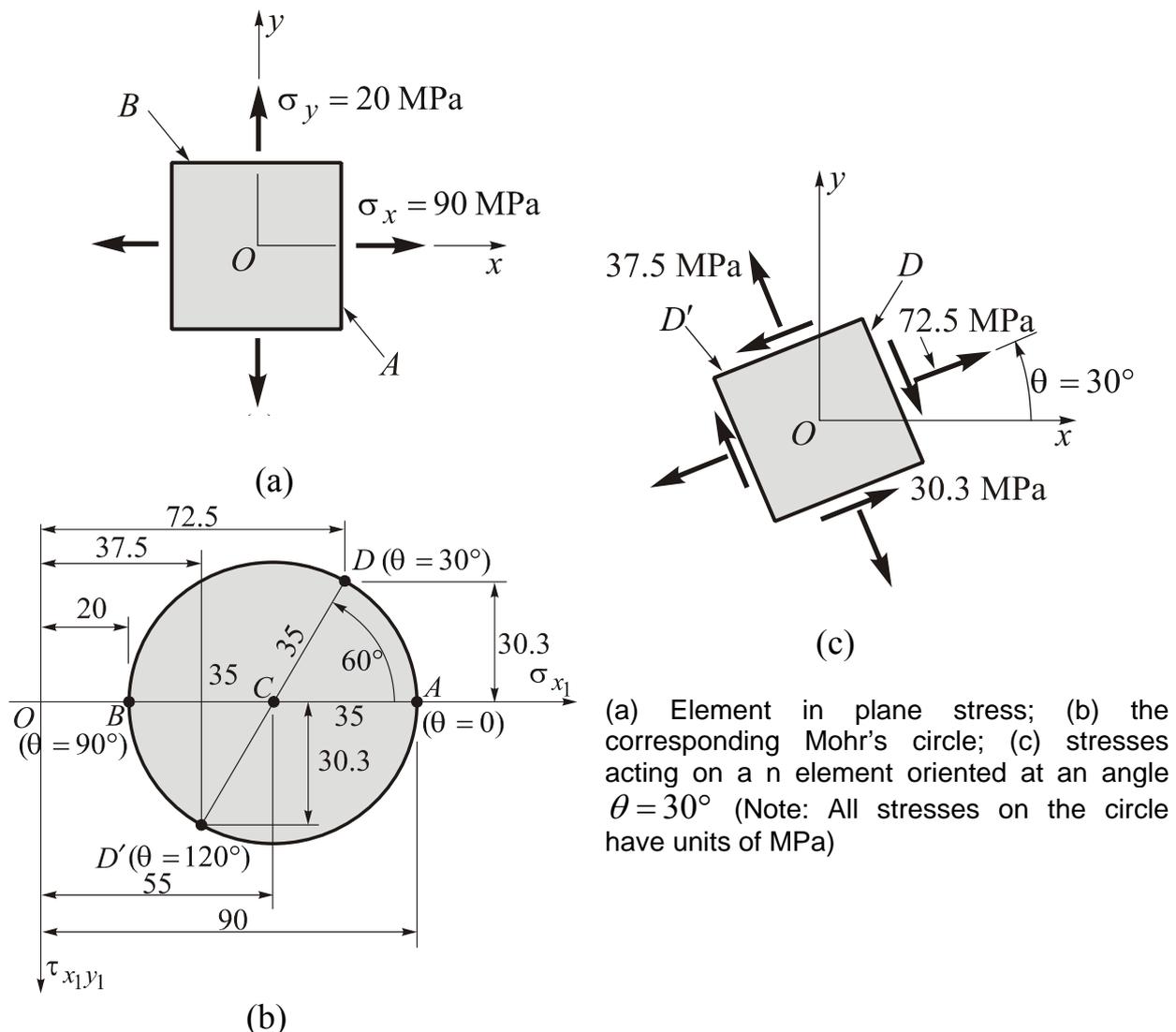
**Note. Mohr's circle eliminates the need to remember the formulas of stress transformation.**



## EXAMPLES

### Example 3.7

At a point on the surface of a cylinder, loaded by internal pressure, the material is subjected to biaxial stresses  $\sigma_x = 90 \text{ MPa}$  and  $\sigma_y = 20 \text{ MPa}$ , as shown on the stress element of figure (a). Using Mohr's circle, determine the stresses acting on an element inclined at an angle  $\theta = 30^\circ$ . (Consider only the in-plane stresses, and show the results on a sketch of a properly oriented element).



**Solution** (1) *Construction of Mohr's circle.* Let us set up the axes for the normal and shear stresses, with  $\sigma_{x_1}$  positive to the right and  $\tau_{x_1y_1}$  positive

downward, as shown in figure (b). Then we place the center  $C$  of the circle on the  $\sigma_{x_1}$  axis at the point where the stress equals the average normal stress:

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} = \frac{90 \text{ MPa} + 20 \text{ MPa}}{2} = 55 \text{ MPa}.$$

Point  $A$ , representing the stresses on the  $x$  face of the element ( $\theta = 0$ ), has coordinates

$$\sigma_{x_1} = 90 \text{ MPa}, \quad \tau_{x_1 y_1} = 0.$$

Similarly, the coordinates of point  $B$ , representing the stresses on the  $y$  face ( $\theta = 90^\circ$ ), are

$$\sigma_{x_1} = 20 \text{ MPa}, \quad \tau_{x_1 y_1} = 0.$$

Now we draw the circle through points  $A$  and  $B$  with center at  $C$  and radius  $R$  equal to

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{90 \text{ MPa} - 20 \text{ MPa}}{2}\right)^2 + 0} = 35 \text{ MPa}.$$

(2) *Stresses on an element inclined at  $\theta = 30^\circ$ .* The stresses acting on a plane oriented at an angle  $\theta = 30^\circ$  are given by the coordinates of point  $D$ , which is at an angle  $2\theta = 60^\circ$  from point  $A$  (see figure (b)). By inspection of the circle, we see that the coordinates of point  $D$  are

$$\begin{aligned}\sigma_{x_1} &= \sigma_{\text{aver}} + R \cos 60^\circ = 55 \text{ MPa} + (35 \text{ MPa})(\cos 60^\circ) = 72.5 \text{ MPa}, \\ \tau_{x_1 y_1} &= -R \sin 60^\circ = -(35 \text{ MPa})(\sin 60^\circ) = -30.3 \text{ MPa}.\end{aligned}$$

In a similar manner, we can find the stresses represented by point  $D'$ , which corresponds to an angle  $\theta = 120^\circ$  (or  $2\theta = 240^\circ$ ):

$$\begin{aligned}\sigma_{x_1} &= \sigma_{\text{aver}} - R \cos 60^\circ = 55 \text{ MPa} - (35 \text{ MPa})(\cos 60^\circ) = 37.5 \text{ MPa}, \\ \tau_{x_1 y_1} &= R \sin 60^\circ = (35 \text{ MPa})(\sin 60^\circ) = 30.3 \text{ MPa}.\end{aligned}$$

These results are shown in figure (c) on a sketch of an element oriented at an angle  $\theta = 30^\circ$ , with all stresses shown in their true directions.

**Note. The sum of the normal stresses on the inclined element is equal to  $\sigma_x + \sigma_y$  or 110 MPa.**

### Example 3.8

An element in plane stress at the surface of a structure is subjected to stresses  $\sigma_x = 100 \text{ MPa}$ ,  $\sigma_y = 35 \text{ MPa}$ , and  $\tau_{xy} = 30 \text{ MPa}$ , as shown in figure (a). Using Mohr's circle, determine the following quantities: (1) the stresses acting on an element inclined at an angle  $\theta = 40^\circ$ , (2) the principal stresses, and (3) the maximum shear stresses. Consider only the in-plane stresses, and show all results on sketches of properly oriented elements.

**Solution** (1) *Construction of Mohr's circle.* Let us set up the axes for Mohr's circle, with  $\sigma_{x_1}$  positive to the right and  $\tau_{x_1y_1}$  positive downward (see figure (b)). The center  $C$  of the circle is located on the  $\sigma_{x_1}$  axis at the point where  $\sigma_{x_1}$  equals the average normal stress:

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} = \frac{100 \text{ MPa} + 35 \text{ MPa}}{2} = 67.5 \text{ MPa}.$$

Point  $A$ , representing the stresses on the  $x$  face of the element ( $\theta = 0$ ), has coordinates

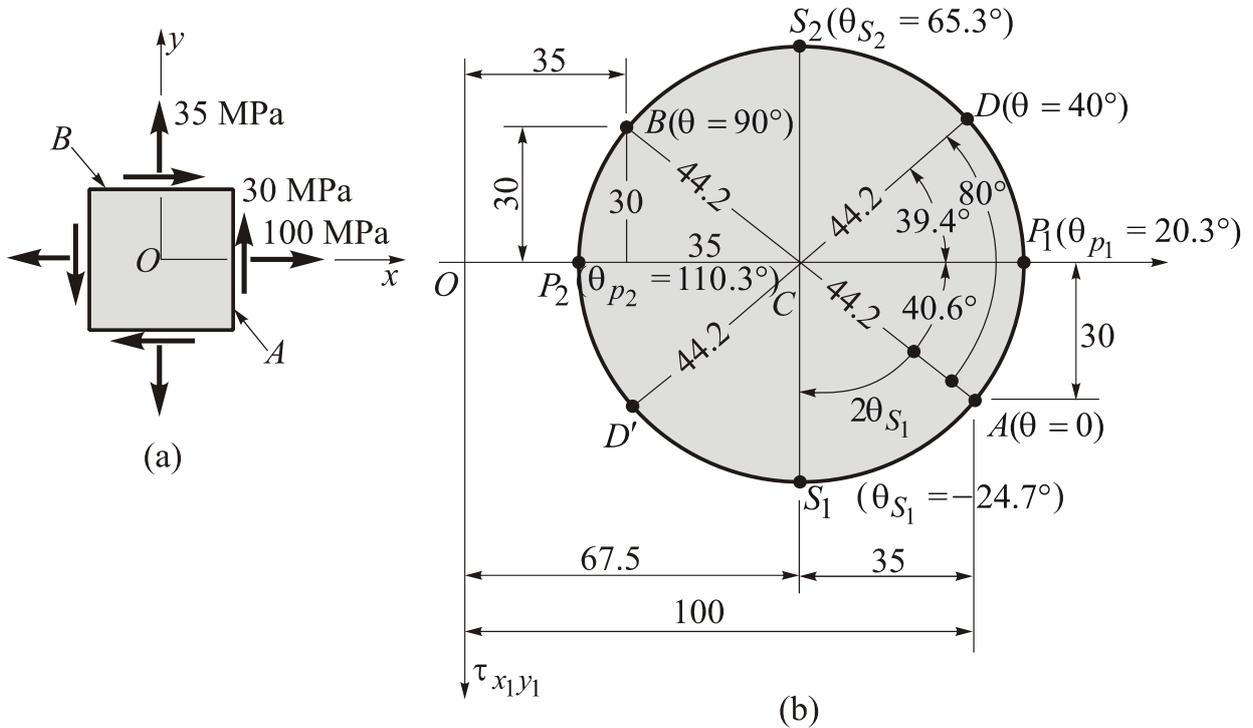
$$\sigma_{x_1} = 100 \text{ MPa}, \quad \tau_{x_1y_1} = 30 \text{ MPa}.$$

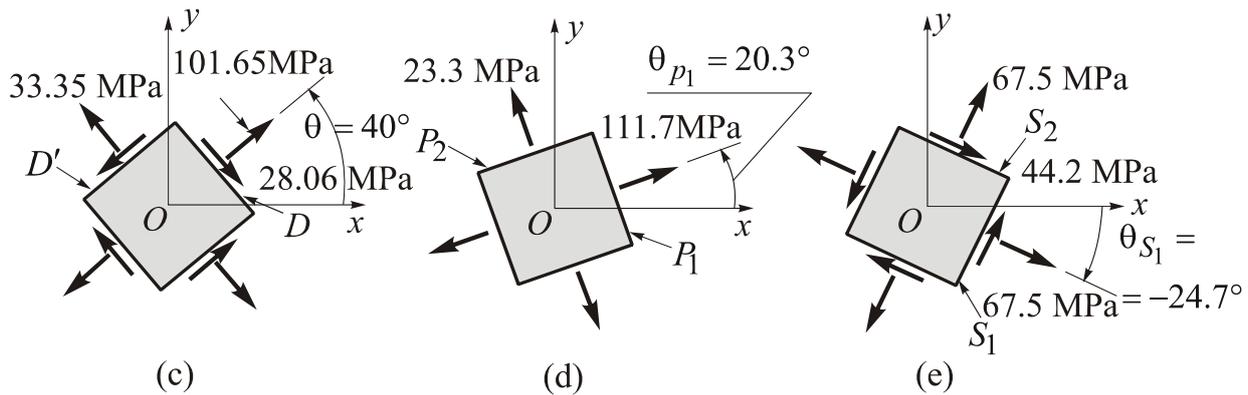
Similarly, the coordinates of point  $B$ , representing the stresses on the  $y$  face ( $\theta = 90^\circ$ ), are

$$\sigma_{x_1} = 35 \text{ MPa}, \quad \tau_{x_1y_1} = -30 \text{ MPa}.$$

The circle is now drawn through points  $A$  and  $B$  with center at  $C$ . The radius of the circle is

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{100 \text{ MPa} - 35 \text{ MPa}}{2}\right)^2 + (30 \text{ MPa})^2} = 44.2 \text{ MPa}.$$





(a) Element in plane stress; (b) the corresponding Mohr's circle; (c) stress acting on an element oriented at  $\theta = 40^\circ$ ; (d) principal stresses; (e) maximum shear stresses

(2) *The stresses acting on a plane oriented at an angle  $\theta = 40^\circ$ .* They are given by the coordinates of point  $D$ , which is at an angle  $2\theta = 80^\circ$  from point  $A$  (see figure (b)). To calculate these coordinates, we need to know the angle between line  $CD$  and the  $\sigma_{x_1}$  axis (that is, angle  $DCP_1$ ), which in turn requires that we know the angle between line  $CA$  and the  $\sigma_{x_1}$  axis (angle  $ACP_1$ ). These angles are found from the geometry of the circle, as follows:

$$\tan \overline{ACP_1} = \frac{30 \text{ MPa}}{35 \text{ MPa}} = 0.857, \quad \overline{ACP_1} = 40.6^\circ,$$

$$\overline{DCP_1} = 80^\circ - \overline{ACP_1} = 80^\circ - 40.6^\circ = 39.4^\circ.$$

Knowing these angles, we can determine the coordinates of point  $D$  directly from the figure:

$$\sigma_{x_1} = 67.5 \text{ MPa} + (44.2 \text{ MPa})(\cos 39.4^\circ) = 101.65 \text{ MPa},$$

$$\tau_{x_1y_1} = -(44.2 \text{ MPa})(\sin 39.4^\circ) = -28.06 \text{ MPa}.$$

In an analogous manner, we can find the stresses represented by point  $D'$ , which corresponds to a plane inclined at an angle  $\theta = 130^\circ$  (or  $2\theta = 260^\circ$ ):

$$\sigma_{x_1} = 67.5 \text{ MPa} - (44.2 \text{ MPa})(\cos 39.4^\circ) = +33.35 \text{ MPa},$$

$$\tau_{x_1y_1} = (44.2 \text{ MPa})(\sin 39.4^\circ) = 28.06 \text{ MPa}.$$

These stresses are shown in figure (c) on a sketch of an element oriented at an angle  $\theta = 40^\circ$  (all stresses are shown in their true directions).

**Note. The sum of the normal stresses is equal to  $\sigma_x + \sigma_y$  or 135 MPa.**

(3) *Principal stresses.* The principal stresses are represented by points  $P_1$  and  $P_2$  on Mohr's circle (see figure (b)). The algebraically larger principal stress (point  $P_1$ ) is

$$\sigma_1 = 67.5 \text{ MPa} + 44.2 \text{ MPa} = 111.7 \text{ MPa},$$

as seen by inspection of the circle. The angle  $2\theta_{P_1}$  to point  $P_1$  from point  $A$  is the angle  $ACP_1$  on the circle, that is,

$$\overline{ACP_1} = 2\theta_{P_1} = 40.6^\circ, \theta_{P_1} = 20.3^\circ.$$

Thus, the plane of the algebraically larger principal stress is oriented at an angle  $\theta_{P_1} = 20.3^\circ$ , as shown in figure (d).

The algebraically smaller principal stress (represented by point  $P_2$ ) is obtained from the circle in a similar manner:

$$\sigma_2 = 67.5 \text{ MPa} - 44.2 \text{ MPa} = 23.3 \text{ MPa}.$$

The angle  $2\theta_{P_2}$  to point  $P_2$  on the circle is  $40.6^\circ + 180^\circ = 220.6^\circ$ ; thus, the second principal plane is defined by the angle  $\theta_{P_2} = 110.3^\circ$ . The principal stresses and principal planes are shown in the figure (d).

**Note. The sum of the normal stresses is equal to 135 MPa.**

(4) *Maximum shear stresses.* The maximum shear stresses are represented by points  $S_1$  and  $S_2$  on Mohr's circle; therefore, the maximum in-plane shear stress (equal to the radius of the circle) is

$$\tau_{\max} = 44.2 \text{ MPa}.$$

The angle  $ACS_1$  from point  $A$  to point  $S_1$  is  $90^\circ - 40.6^\circ = 49.4^\circ$ , and therefore the angle  $2\theta_{S_1}$ , for point  $S_1$  is

$$2\theta_{S_1} = -49.4^\circ.$$

This angle is negative because it is measured clockwise on the circle. The corresponding angle  $\theta_{S_1}$  to the plane of the maximum positive shear stress is one-half that value, or  $\theta_{S_1} = -24.7^\circ$ , as shown in Figs. (b) and (e). The maximum negative shear stress (point  $S_2$  on the circle) has the same numerical value as the maximum positive stress (44.2 MPa).

The normal stresses acting on the planes of maximum shear stress are equal to  $\sigma_{aver}$ , which is the abscissa of the center  $C$  of the circle (67.5 MPa). These stresses are also shown in figure (e).

**Note. The planes of maximum shear stresses are oriented at  $45^\circ$  to the principal planes.**

**Example 3.9**

At a point on the surface of a shaft the stresses are  $\sigma_x = -50$  MPa,  $\sigma_y = 10$  MPa, and  $\tau_{xy} = -40$  MPa, as shown in figure (a). Using Mohr's circle, determine the following quantities: (1) the stresses acting on an element inclined at an angle  $\theta = 45^\circ$ , (2) the principal stresses, and (3) the maximum shear stresses.

**Solution** (1) *Construction of Mohr's circle.* The axes for the normal and shear stresses in the Mohr's circle are shown in figure (b), with  $\sigma_{x_1}$  positive to the right and  $\tau_{x_1y_1}$  positive downward. The center  $C$  of the circle is located on the  $\sigma_{x_1}$  axis at the point where the stress equals the average normal stress:

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} = \frac{-50 \text{ MPa} + 10 \text{ MPa}}{2} = -20 \text{ MPa} .$$

Point  $A$ , representing the stresses on the  $x$  face of the element ( $\theta = 0$ ), has coordinates

$$\sigma_{x_1} = -50 \text{ MPa} , \quad \tau_{x_1y_1} = -40 \text{ MPa} .$$

Similarly, the coordinates of point  $B$ , representing the stresses on the  $y$  face ( $\theta = 90^\circ$ ), are

$$\sigma_{x_1} = 10 \text{ MPa} , \quad \tau_{x_1y_1} = 40 \text{ MPa} .$$

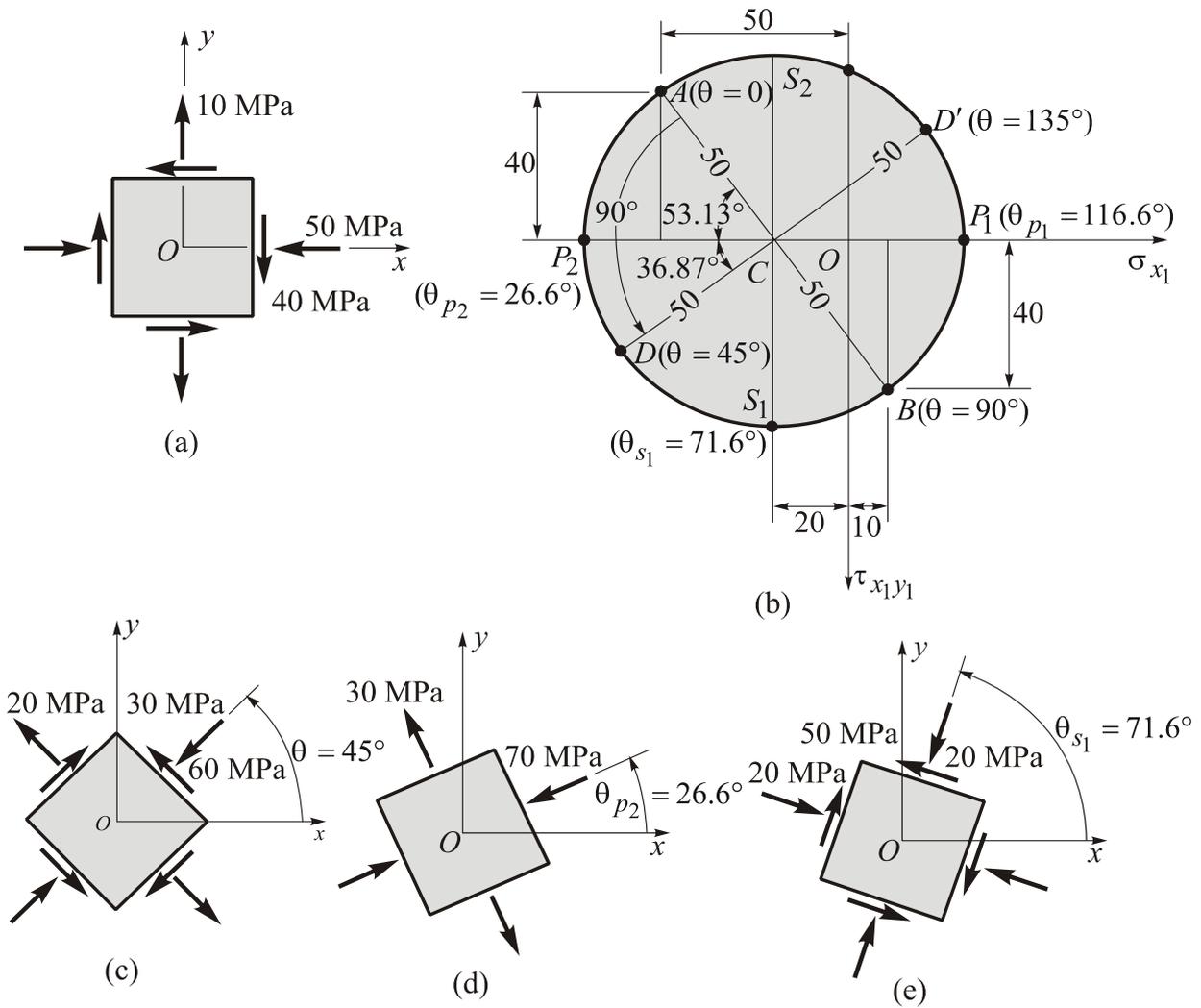
The circle is now drawn through points  $A$  and  $B$  with center at  $C$  and radius  $R$  equal to:

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{-50 \text{ MPa} - 10 \text{ MPa}}{2}\right)^2 + (-40 \text{ MPa})^2} = 50 \text{ MPa} .$$

(2) *Stresses on an element inclined at  $\theta = 45^\circ$ .* These stresses are given by the coordinates of point  $D$ , which is at an angle  $2\theta = 90^\circ$  from point  $A$  (figure (b)). To evaluate these coordinates, we need to know the angle between line  $CD$  and the negative  $\sigma_{x_1}$  axis (that is, angle  $DCP_2$ ), which in turn requires that we know the angle between line  $CA$  and the negative  $\sigma_{x_1}$  axis (angle  $ACP_2$ ). These angles are found from the geometry of the circle as follows:

$$\tan \overline{ACP_2} = \frac{40 \text{ MPa}}{30 \text{ MPa}} = \frac{4}{3}, \quad \overline{ACP_2} = 53.13^\circ ,$$

$$\overline{DCP_2} = 90^\circ - \overline{ACP_2} = 90^\circ - 53.13^\circ = 36.87^\circ .$$



(a) Element in plane stress; (b) the corresponding Mohr's circle; (c) stresses acting on an element oriented at  $\theta=45^\circ$ ; (d) principal stresses, and (e) maximum shear stresses. (Note: All stresses on the circle have units of MPa)

Knowing these angles, we can obtain the coordinates of point  $D$  directly from the figure:

$$\sigma_{x_1} = -20 \text{ MPa} - (50 \text{ MPa})(\cos 36.87^\circ) = -60 \text{ MPa},$$

$$\tau_{x_1y_1} = (50 \text{ MPa})(\sin 36.87^\circ) = 30 \text{ MPa}.$$

In an analogous manner, we can find the stresses represented by point  $D'$ , which corresponds to a plane inclined at an angle  $\theta = 135^\circ$  (or  $2\theta = 270^\circ$ ):

$$\sigma_{x_1} = -20 \text{ MPa} + (50 \text{ MPa})(\cos 36.87^\circ) = 20 \text{ MPa},$$

$$\tau_{x_1y_1} = (-50 \text{ MPa})(\sin 36.87^\circ) = -30 \text{ MPa}.$$

These stresses are shown in Fig. c on a sketch of an element oriented at an angle  $\theta = 45^\circ$  (all stresses are shown in their true directions).

**Note. The sum of the normal stresses is equal to  $\sigma_x + \sigma_y$  or  $-40$  MPa.**

(3) *Principal stresses.* They are represented by points  $P_1$  and  $P_2$  on Mohr's circle. The algebraically larger principal stress (represented by point  $P_1$ ) is

$$\sigma_1 = -20 \text{ MPa} + 50 \text{ MPa} = 30 \text{ MPa},$$

as seen by inspection of the circle. The angle  $2\theta_{p_1}$  to point  $P_1$  from point  $A$  is the angle  $ACP_1$  measured counterclockwise on the circle, that is,

$$\overline{ACP_1} = 2\theta_{p_1} = 53.13^\circ + 180^\circ = 233.13^\circ, \quad \theta_{p_1} = 116.6^\circ.$$

Thus, the plane of the algebraically larger principal stress is oriented at an angle  $\theta_{p_1} = 116.6^\circ$ .

The algebraically smaller principal stress (point  $P_2$ ) is obtained from the circle in a similar manner:

$$\sigma_3 = -20 \text{ MPa} - 50 \text{ MPa} = -70 \text{ MPa}.$$

The angle  $2\theta_{p_2}$  to point  $P_2$  on the circle is  $53.13^\circ$ . The second principal plane is defined by the angle  $2\theta_{p_2} = 26.6^\circ$ .

The principal stresses and principal planes are shown in Fig. (d).

**Note. The sum of the normal stresses is equal to  $\sigma_x + \sigma_y$  or  $-40$  MPa.**

(4) *Maximum shear stresses.* The maximum positive and negative shear stresses are represented by points  $S_1$  and  $S_2$  on Mohr's circle (figure (b)). Their magnitudes, equal to the radius of the circle, are

$$\tau_{\max} = 50 \text{ MPa}.$$

The angle  $ACS_1$  from point  $A$  to point  $S_1$  is  $90^\circ + 53.13^\circ = 143.13^\circ$ , and therefore the angle  $2\theta_{s_1}$  for point  $S_1$  is

$$2\theta_{s_1} = 143.13^\circ.$$

The corresponding angle  $\theta_{s_1}$  to the plane of the maximum positive shear stress is one-half that value, or  $\theta_{s_1} = 71.6^\circ$ , as shown in figure (e). The maximum negative shear stress (point  $S_2$  on the circle) has the same numerical value as the positive stress (50 MPa).

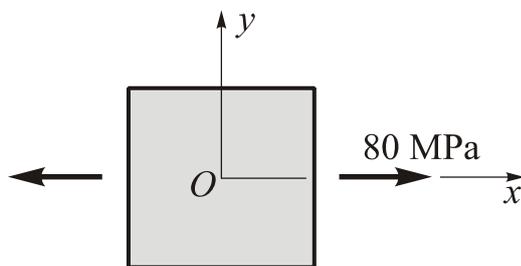
The normal stresses acting on the planes of maximum shear stress are equal to  $\sigma_{aver}$ , which is the coordinate of the center  $C$  of the circle ( $-20$  MPa). These stresses are also shown in figure (e).

**Note. The planes of maximum shear stress are oriented at  $45^\circ$  to the principal planes.**

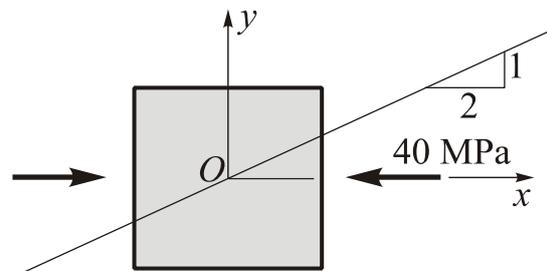


## PROBLEMS

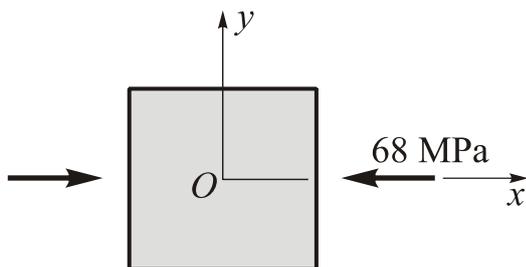
**Problem 3.78** An element in uniaxial stress is subjected to tensile stresses  $\sigma_x = 80 \text{ MPa}$ , as shown in the figure. Using Mohr's circle, determine: (1) the stresses acting on an element oriented at an angle  $\theta = 21.8^\circ$  from the  $x$  axis, and (2) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.



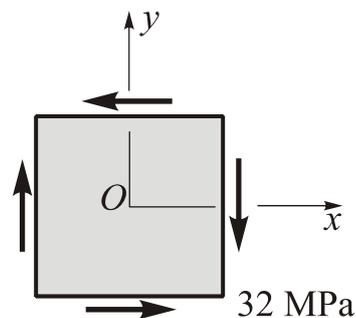
**Problem 3.80** An element in uniaxial stress is subjected to a compressive stress of  $40 \text{ MPa}$ , as shown in the figure. Using Mohr's circle, determine: (1) the stresses acting on an element oriented at a slope of 1 on 2 (see figure), and (2) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.



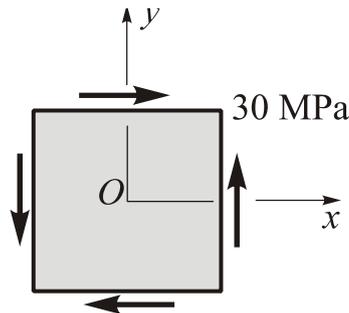
**Problem 3.79** An element in uniaxial stress is subjected to compressive stresses  $\sigma_x = 68 \text{ MPa}$ , as shown in the figure. Using Mohr's circle, determine: (1) the stresses acting on an element oriented at an angle  $\theta = 31^\circ$  from the  $x$  axis, and (2) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.



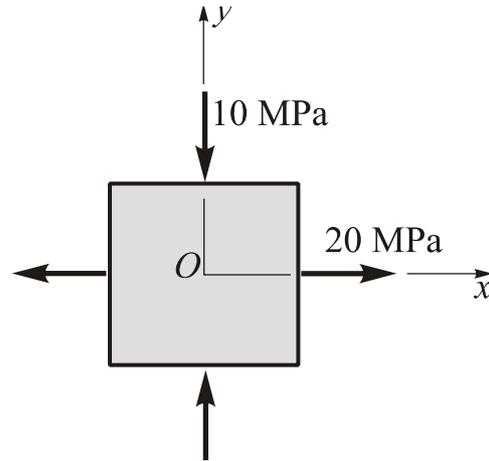
**Problem 3.81** An element in pure shear is subjected to stresses  $\tau_{xy} = -32 \text{ MPa}$ , as shown in the figure. Using Mohr's circle, determine: (1) the stresses acting on an element oriented at an angle  $\theta = 20^\circ$  from the  $x$  axis, and (2) the principal stresses. Show all results on sketches of properly oriented elements.



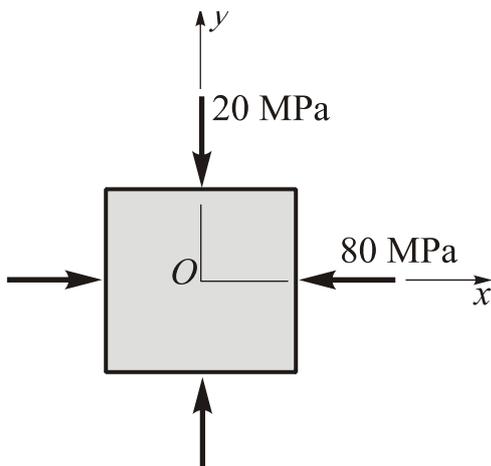
**Problem 3.82** An element in pure shear is subjected to stresses  $\tau_{xy} = 30$  MPa, as shown in the figure. Using Mohr's circle, determine: (1) the stresses acting on an element oriented at an angle  $\theta = 75^\circ$  from the  $x$  axis, and (2) the principal stresses. Show all results on sketches of properly oriented elements.



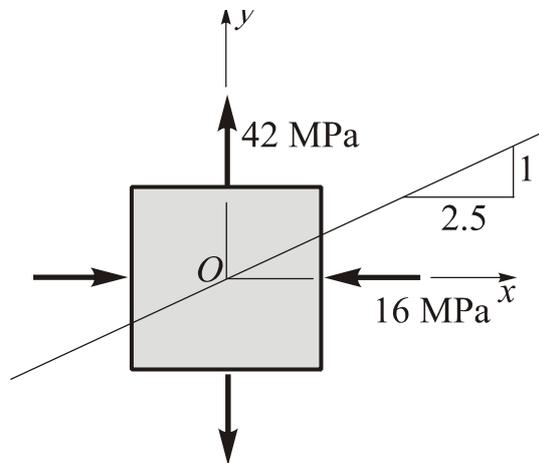
shown in the figure. Using Mohr's circle, determine: (1) the stresses acting on an element oriented at an angle  $\theta = 60^\circ$  from the  $x$  axis, and (2) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.



**Problem 3.83** An element in biaxial stress is subjected to stresses  $\sigma_x = -80$  MPa and  $\sigma_y = -20$  MPa, as shown in the figure. Using Mohr's circle, determine: (1) the stresses acting on an element oriented at an angle  $\theta = 30^\circ$  from the  $x$  axis, and (2) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.

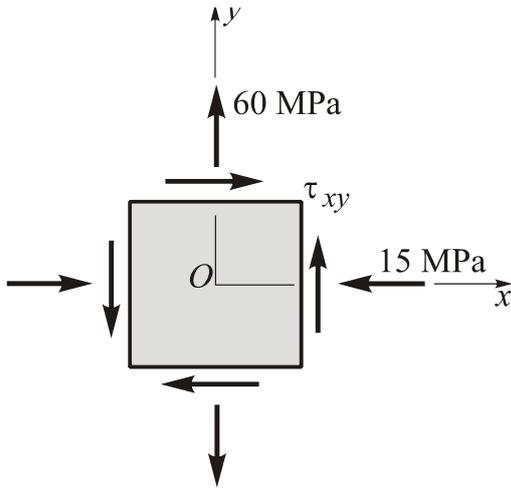


**Problem 3.85** An element in biaxial stress is subjected to stresses  $\sigma_x = -16$  MPa and  $\sigma_y = 42$  MPa, as shown in the figure. Using Mohr's circle, determine: (1) the stresses acting on an element oriented at a slope of 1 on 2.5 (see figure), and (2) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.

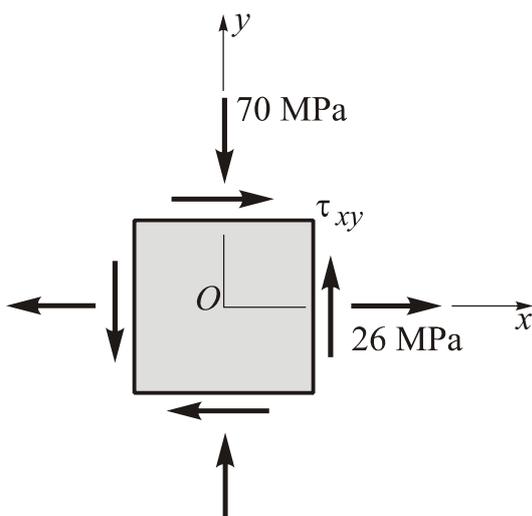


**Problem 3.84** An element in biaxial stress is subjected to stresses  $\sigma_x = 20$  MPa and  $\sigma_y = -10$  MPa, as

**Problem 3.86** For an element in plane stress the normal stresses are  $\sigma_x = -15$  MPa and  $\sigma_y = 60$  MPa (see figure). Using Mohr's circle, determine the maximum permissible value of the shear stress  $\tau_{xy}$  if the allowable shear stress in the material is 65 MPa.

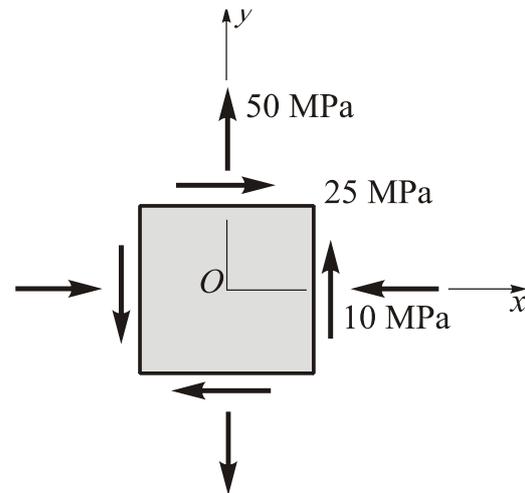


**Problem 3.87** For an element in plane stress, the normal stresses are  $\sigma_x = 26$  MPa and  $\sigma_y = -70$  MPa (see figure). Using Mohr's circle, determine the maximum permissible value of the shear stress  $\tau_{xy}$  if the allowable shear stress in the material is 55 MPa.

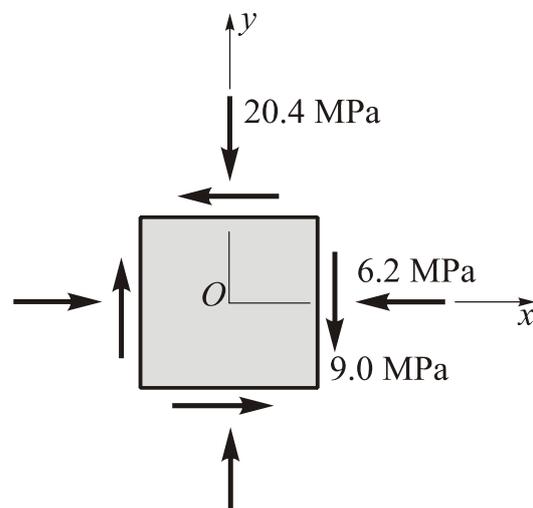


**Problem 3.88** An element in plane stress is subjected to stresses

$\sigma_x = -10$  MPa,  $\sigma_y = 50$  MPa, and  $\tau_{xy} = 25$  MPa, as shown in the figure. Using Mohr's circle, determine the stresses acting on an element oriented at an angle  $\theta = 45^\circ$  from the  $x$  axis. Show these stresses on a sketch of an element oriented at the angle  $\theta$ .



**Problem 3.89** An element in plane stress is subjected to stresses  $\sigma_x = -6.2$  MPa,  $\sigma_y = -20.4$  MPa, and  $\tau_{xy} = -9$  MPa, as shown in the figure. Using Mohr's circle, determine the stresses acting on an element oriented at an angle  $\theta = 30^\circ$  from the  $x$  axis. Show these stresses on a sketch of an element oriented at the angle  $\theta$ .



### 3.5 Hooke's Law for Plane Stress and its Special Cases. Change of Volume. Relations between $E$ , $G$ , and $\nu$

#### 3.5.1 Hooke's Law for Plane Stress

In Section 3.1 the stresses acting on inclined planes when linearly elastic material is subjected to plane stress (Fig. 3.22) were considered. The stress-transformation equations derived in those calculations were obtained solely from equilibrium, and therefore the properties of the materials were not needed. Now, it is important to investigate the *strains* in the material, which means that the material properties must be considered. However, we will limit our consideration to materials that meet two important conditions: first, the *material is uniform throughout the body and has the same properties in all directions* (homogeneous and isotropic material), and second, the *material follows Hooke's law* (linearly elastic material). Under these conditions, we can obtain the relationships between the stresses and strains in the body.

Let us begin by considering the normal strains  $\epsilon_x$ ,  $\epsilon_y$ , and  $\epsilon_z$  in plane stress. The effects of these strains are shown in Fig. 3.23, which represents the changes in dimensions of small element having edges of lengths  $a$ ,  $b$ , and  $c$ . All three strains are shown positive (elongation) in Fig. 3.23. The strains can be expressed in terms of the stresses (Fig. 3.22) by superimposing the effects of the individual stresses. For instance, the strain  $\epsilon_x$  in the  $x$  direction due to the stress  $\sigma_x$  is equal to  $\sigma_x/E$ , where  $E$  is the modulus of elasticity. Also, the strain  $\epsilon_x$  due to the stress  $\sigma_y$  is equal to  $-\nu\sigma_y/E$ , where  $\nu$  is Poisson's ratio. Of course, we will assume that as earlier, the *shear stress  $\tau_{xy}$  produces no normal strains* in the  $x$ ,  $y$ , or  $z$  directions. Thus, the resultant strain in the  $x$  direction is

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y). \quad (3.47)$$

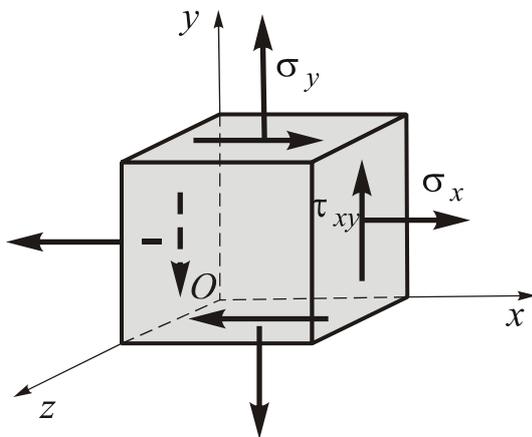


Fig. 3.22 Element of material in plane stress ( $\sigma_z = 0$ )

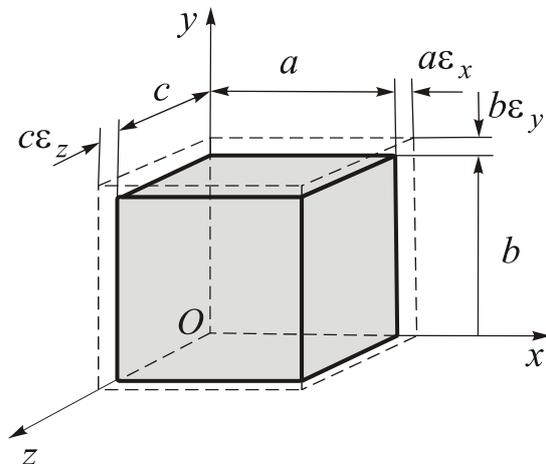


Fig. 3.23 Element of material subjected to normal strains  $\epsilon_x$ ,  $\epsilon_y$ , and  $\epsilon_z$

Similarly, we obtain the strains in the  $y$  and  $z$  directions:

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x), \quad (3.48)$$

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y). \quad (3.49)$$

These equations may be used to find the normal strains (in plane stress) when the stresses are known.

The *shear stress*  $\tau_{xy}$  (Fig. 3.22) causes a *distortion* of the element such that each  $z$  face becomes a rhombus (Fig. 3.24). The shear strain  $\gamma_{xy}$  is the decrease in angle between the  $x$  and  $y$  faces of the element and is related to the shear stress by Hooke's law in shear, as follows:

$$\gamma_{xy} = \frac{\tau_{xy}}{G}, \quad (3.50)$$

where  $G$  is the shear modulus of elasticity.

**Note.** We assume, that the normal stresses  $\sigma_z$  and  $\sigma_y$  have no effect on the shear strain  $\gamma_{xy}$ .

Consequently, Eqs. (3.47 through 3.50) give the strains (in plane stress) when all stresses ( $\sigma_x$ ,  $\sigma_y$  and  $\tau_{xy}$ ) act simultaneously.

The first two equations (3.47 and 3.48) give the strains  $\varepsilon_x$  and  $\varepsilon_y$  in terms of the stresses. These equations can be solved simultaneously for the stresses in terms of the strains:

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y), \quad (3.51)$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x). \quad (3.52)$$

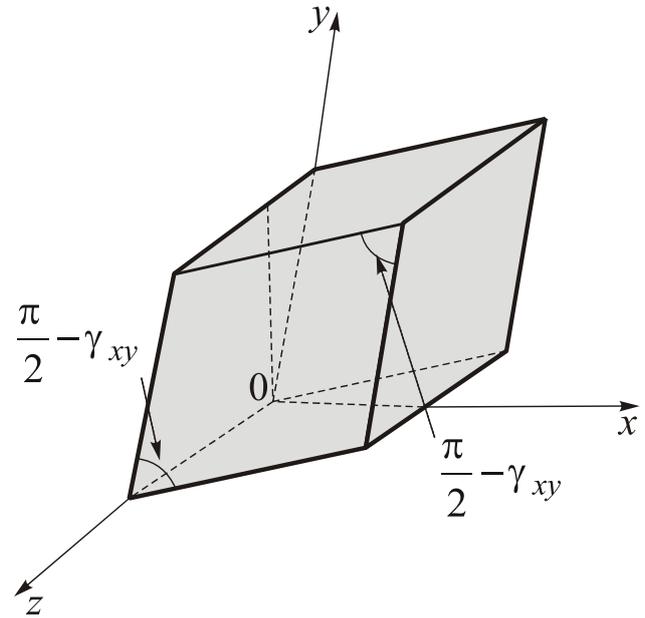


Fig. 3.24 Shear strain  $\gamma_{xy}$  and distortion of  $z$ -face

In addition, we have the following equation for the shear stress in terms of the shear strain:

$$\tau_{xy} = G\gamma_{xy}. \quad (3.53)$$

Equations (3.51) through (3.53) may be used to find the stresses (in plane stress) when the strains are known. Of course, the normal stress  $\sigma_z$  in the  $z$  direction is equal to zero.

Equations (3.47) through (3.53) are known collectively as *Hooke's law for plane stress*. They contain three material constants ( $E$ ,  $G$ , and  $\nu$ ) but only two are independent because of the relationship

$$G = \frac{E}{2(1+\nu)} \quad (3.54)$$

It will be founded below.

### 3.5.2 Special Cases of Hooke's Law

In the special case of *biaxial stress* (Fig. 3.12), we have  $\tau_{xy} = 0$  and therefore Hooke's law for plane stress simplifies to

$$\begin{aligned} \varepsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y), \\ \varepsilon_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x), \\ \varepsilon_z &= -\frac{\nu}{E}(\sigma_x + \sigma_y), \end{aligned} \quad (3.55)$$

or

$$\begin{aligned} \sigma_x &= \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y), \\ \sigma_y &= \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x). \end{aligned} \quad (3.56)$$

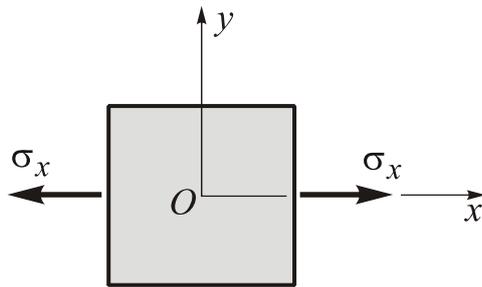


Fig. 3.25 Uniaxial stress state

These equations are the same as Eqs. (3.47) through (3.49) and (3.51), (3.52) because the effects of normal and shear stresses are independent of each other.

For uniaxial stress, with  $\sigma_y = 0$  (Fig. 3.25), the equations of Hooke's law simplify even further:

$$\varepsilon_x = \frac{\sigma_x}{E}, \quad (3.57)$$

$$\varepsilon_y = \varepsilon_z = -\frac{\nu\sigma_x}{E}, \quad (3.58)$$

$$\sigma_x = E\varepsilon_x. \quad (3.59)$$

Finally, we consider *pure shear* (Fig. 3.26), which means that  $\sigma_x = \sigma_y = 0$ .

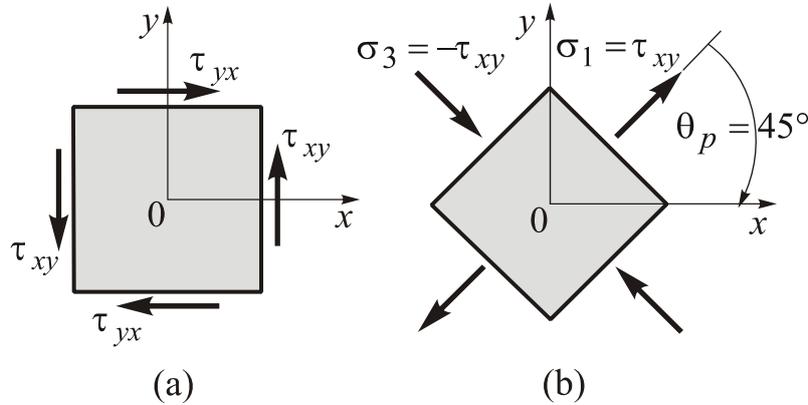


Fig. 3.26 Element in pure shear

Then we obtain

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = 0, \quad (3.60)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}. \quad (3.61)$$

**Note, that the normal stress  $\sigma_z$  is equal to zero in all three of these special cases of plane stress state.**

### 3.5.3 Change of Volume

Similarly to Section 2.1.3, the unit volume change at a point in a strained body can be found by considering the deformed element of Fig. 3.23. The original volume of this element is  $V_0 = abc$ , and its final volume is

$$V_f = abc(1 + \varepsilon_x)(1 + \varepsilon_y)(1 + \varepsilon_z), \quad \text{or} \quad (3.62)$$

$$V_f = abc(1 + \varepsilon_x + \varepsilon_y + \varepsilon_z + \varepsilon_x\varepsilon_y + \varepsilon_x\varepsilon_z + \varepsilon_x\varepsilon_y\varepsilon_z). \quad (3.63)$$

If we only consider structures having very small strains, then we can disregard the products of small strains in comparison with the strains themselves. Thus, the final volume (Eq. 3.63) becomes

$$V_f = abc(1 + \varepsilon_x + \varepsilon_y + \varepsilon_z). \quad (3.64)$$

Therefore, the absolute change in volume is

$$\Delta V = V_f - V_0 = abc(\varepsilon_x + \varepsilon_y + \varepsilon_z), \quad (3.65)$$

and the unit (relative) volume change (or dilatation) becomes

$$e = \frac{\Delta V}{V_0} = \varepsilon_x + \varepsilon_y + \varepsilon_z. \quad (3.66)$$

This equation gives the dilatation in terms of the normal strains.

**Note. (1) Formula (3.66) is valid for any material, whether or not it follows Hooke's law, provided the strains are small. (2) This equation is not limited to plane stress, but is valid for any stress conditions. (3) Shear strains produce no change in volume.**

When the material follows Hooke's law, we can substitute Eqs. ((3.47) through (3.49)) into Eq. (3.66) and obtain the following expression for the unit volume change (in plane stress) in terms of the stresses:

$$e = \frac{\Delta V}{V_0} = \frac{1-2\nu}{E}(\sigma_x + \sigma_y). \quad (3.67)$$

Knowing this expression for  $e$ , we can find the volume change for any object subjected to plane stress by integrating throughout its volume.

### 3.5.4 Relations between $E$ , $G$ , and $\nu$

Now let us develop the fundamental relationship, Eq. (3.54), connecting the modulus of elasticity  $E$ , shear modulus of elasticity  $G$ , and Poisson's ratio  $\nu$ . Consider an element subjected to pure shear. Mohr's circle for this case shows that  $\sigma_1 = \tau_{\max}$  and  $\sigma_3 = -\tau_{\max}$  on the planes making  $45^\circ$  with the shear planes (see Fig. 3.20a).

Thus application of the Eq. (3.47) for  $\varepsilon_x = \varepsilon_1$ ,  $\sigma_x = \sigma_1$ ,  $\sigma_y = \sigma_3$ , and  $\tau_{xy} = 0$  results in

$$\varepsilon_1 = \frac{\sigma_1}{E} - \nu \frac{\sigma_3}{E} = \frac{\tau_{\max}}{E}(1 + \nu). \quad (3.68)$$

On the other hand, for the state of pure shear strain, it is observed from Mohr's circle that  $\varepsilon_1$  is  $\gamma_{\max} / 2$  (see Chapter 5). Hooke's law connects the shear strain and shear stress:  $\gamma_{\max} = \tau_{\max} / G$ . Hence

$$\varepsilon_1 = \frac{\tau_{\max}}{2G}. \quad (3.69)$$

Finally, equating the alternative expressions for  $\varepsilon_1$  in Eqs. (3.68) and (3.69), we obtain

$$G = \frac{E}{2(1 + \nu)}. \quad (3.70)$$

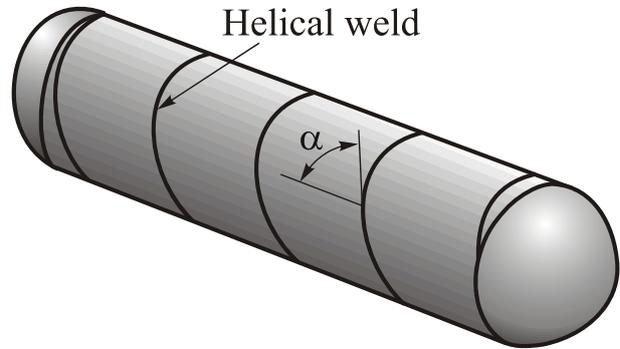
It can be shown that for any choice of orientation of coordinate axes  $x$  and  $y$ , the same result Eq. (3.70) is obtained. Therefore, the two- and three-dimensional stress-strain relations for an isotropic and elastic material can be written in terms of two constants.



## EXAMPLES

### Example 3.10

A cylindrical pressure vessel is constructed from a long, narrow steel plate by wrapping the plate around a mandrel and then welding along the edges of the plate to make a helical joint (see figure (a)). The helical weld makes an angle  $\alpha = 55^\circ$  with the longitudinal axis. The vessel has inner radius  $r = 1.8$  m and wall thickness  $t = 20$  mm. The material is steel with modulus  $E = 200$  GPa and Poisson's ratio  $\nu = 0.30$ . The internal pressure  $p$  is 800 kPa. Calculate the following quantities for the cylindrical part of the vessel: (1) the circumferential and longitudinal stresses  $\sigma_1$  and  $\sigma_2$ , respectively; (2) the maximum in-plane and out-of-plane shear stresses; (3) the circumferential and longitudinal strains  $\varepsilon_1$  and  $\varepsilon_2$ , respectively; and (4) the normal stress  $\sigma_w$  and shear stress  $\tau_w$  acting perpendicular and parallel, respectively, to the welded seam.



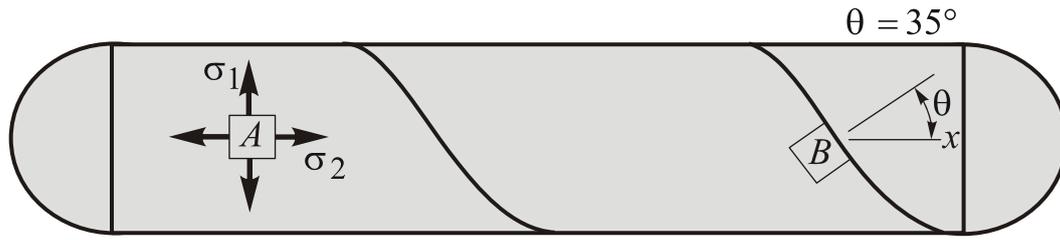
(a) Cylindrical pressure vessel with a helical weld

**Solution** (1) *Circumferential and longitudinal stresses.* The circumferential and longitudinal stresses  $\sigma_1$  and  $\sigma_2$ , respectively, are pictured in Fig. b, where they are shown acting on a stress element at point  $A$  on the wall of the vessel. The magnitudes of the stresses can be calculated from equations of equilibrium:

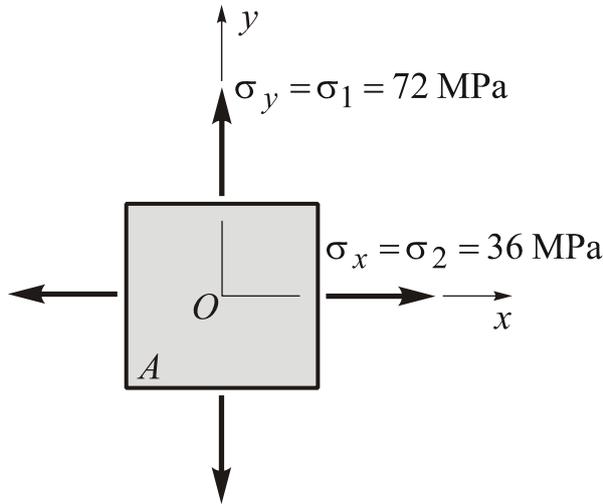
$$\sigma_1 = \frac{pr}{t}, \quad \sigma_2 = \frac{pr}{2t}, \quad \text{or}$$

$$\sigma_1 = \frac{(800 \text{ kPa})(1.8 \text{ m})}{20 \text{ mm}} = 72 \text{ MPa}, \quad \sigma_2 = \frac{\sigma_1}{2} = 36 \text{ MPa}.$$

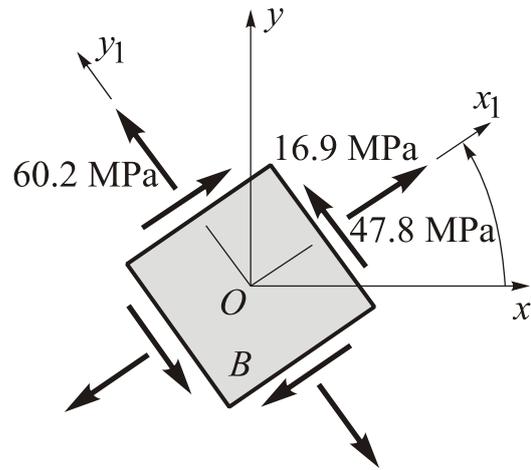
The stress element at point  $A$  is shown again in Fig. c, where the  $x$  axis is in the longitudinal direction of the cylinder and the  $y$  axis is in the circumferential direction. Since there is no stress in the  $z$  direction ( $\sigma_3 = 0$ ), the element is in biaxial stress. *This assumption is based on the comparison of magnitudes of internal pressure and the smaller of two principal stresses, i.e.  $\sigma_2$ .*



(b) Side view of pressure vessel



(c) Two-dimensional view of an element in biaxial stress oriented to  $xy$  axes



(d) Two-dimensional view of an element oriented to the  $x_1y_1$  axes

The ratio of the internal pressure (800 kPa) to the smaller in-plane principal stress (36 MPa) is 0.022. Therefore, our assumption that we may disregard any stresses in the  $z$  direction and consider all elements in the cylindrical shell, even those at the inner surface, to be in biaxial stress is justified.

(2) *Maximum shear stresses.* The largest in-plane shear stresses occur on planes that are rotated  $45^\circ$  about the  $z$  axis (see Eq. 3.39):

$$(\tau_{\max})_z = \frac{\sigma_1 - \sigma_2}{2} = \frac{\sigma_1}{4} = \frac{pr}{4t} = 18 \text{ MPa} .$$

The maximum out-of-plane shear stresses are obtained by  $45^\circ$  rotations about the  $x$  and  $y$  axes, respectively, thus,

$$(\tau_{\max})_x = \frac{\sigma_1}{2} = \frac{pr}{2t}, \quad (\tau_{\max})_y = \frac{\sigma_2}{2} = \frac{pr}{4t} .$$

Because we are disregarding the normal stress in the  $z$  direction, the largest out-of-plane shear stress equals to:

$$\tau_{\max} = \frac{\sigma_1}{2} = \frac{pr}{2t} = 36 \text{ MPa} .$$

This last stress is the absolute maximum shear stress in the wall of the vessel.

(3) *Circumferential and longitudinal strains.* Since the largest stresses are well below the yield stress of steel, we may assume that Hooke's law applies to the wall of the vessel. Then we can obtain the strains in the  $x$  and  $y$  directions (figure (c)) from equations of the Hooke's law for biaxial stress (3.55):

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y), \quad \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x).$$

Note, that the strain  $\varepsilon_x$  is the same as the principal strain  $\varepsilon_2$  in the longitudinal direction and that the strain  $\varepsilon_y$  is the same as the principal strain  $\varepsilon_1$  in the circumferential direction. Also, the stress  $\sigma_x$  is the same as the stress  $\sigma_2$ , and the stress  $\sigma_y$  is the same as the stress  $\sigma_1$ . The preceding two equations can be written in the following forms:

$$\varepsilon_2 = \frac{\sigma_2}{E}(1 - 2\nu) = \frac{pr}{2tE}(1 - 2\nu),$$

$$\varepsilon_1 = \frac{\sigma_1}{2E}(2 - \nu) = \frac{pr}{2tE}(2 - 2\nu).$$

Substituting numerical values, we find

$$\varepsilon_2 = \frac{\sigma_2}{E}(1 - 2\nu) = \frac{(36 \text{ MPa})[1 - 2(0.30)]}{200 \text{ GPa}} = 72 \times 10^{-6},$$

$$\varepsilon_1 = \frac{\sigma_1}{2E}(2 - \nu) = \frac{(72 \text{ MPa})(2 - 0.30)}{2(200 \text{ GPa})} = 306 \times 10^{-6}.$$

These are the longitudinal and circumferential strains, respectively, in the cylinder.

(4) *Normal and shear stresses acting on the welded seam.* The stress element at point  $B$  in the wall of the cylinder (figure (d)) is oriented so that its sides are parallel and perpendicular to the weld. The angle  $\theta$  for the element is

$$\theta = 90^\circ - \alpha = 35^\circ,$$

as shown in figure (d). The stress-transformation equations may be used to obtain the normal and shear stresses acting on the side faces of this element.

The normal stress  $\sigma_{x_1}$  and the shear stress  $\tau_{x_1y_1}$  acting on the  $x_1$  face of the element are obtained from Eqs. (3.8) and (3.9), which are repeated here:

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta,$$

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta.$$

Substituting  $\sigma_x = \sigma_2 = pr / 2t$ ,  $\sigma_y = \sigma_1 = pr / t$  and  $\tau_{xy} = 0$ , we obtain

$$\sigma_{x_1} = \frac{pr}{4t}(3 - \cos 2\theta), \quad \tau_{x_1y_1} = \frac{pr}{4t} \sin 2\theta.$$

These equations give the normal and shear stresses acting on an inclined plane oriented at an angle  $\theta$  with the longitudinal axis of the cylinder.

Substituting  $pr / 4t = 18 \text{ MPa}$  and  $\theta = 35^\circ$  into last two equations, we obtain

$$\sigma_{x_1} = 47.8 \text{ MPa}, \quad \tau_{x_1y_1} = 16.9 \text{ MPa}.$$

These stresses are shown on the stress element of Fig. d.

To complete the stress analysis, we can calculate the normal stress  $\sigma_{y_1}$  acting on the  $y_1$  face of the element from the sum of the normal stresses on perpendicular faces:

$$\sigma_1 + \sigma_2 = \sigma_{x_1} + \sigma_{y_1}.$$

Substituting numerical values, we get

$$\sigma_{y_1} = \sigma_1 + \sigma_2 - \sigma_{x_1} = 72 \text{ MPa} + 36 \text{ MPa} - 47.8 \text{ MPa} = 60.2 \text{ MPa}$$

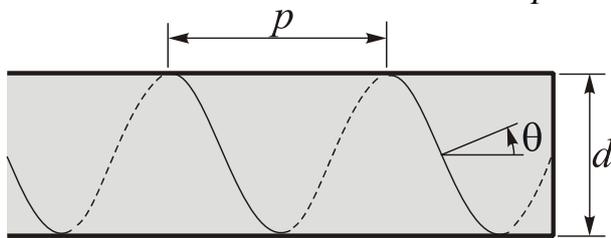
as shown in figure (d).

From the figure, we see that the normal and shear stresses acting perpendicular and parallel, respectively, to the welded seam are

$$\sigma_w = 47.8 \text{ MPa}, \quad \tau_w = 16.9 \text{ MPa}.$$

Interestingly, when seen in a side view, a helix follows the shape of a sine curve (Fig. e). The pitch of the helix is

$$p = \pi d \tan \theta,$$



where  $d$  is the diameter of the circular cylinder and  $\theta$  is the angle between a normal to the helix and a longitudinal line. The width of the plate that wraps into the cylindrical shape is

$$w = \pi d \sin \theta.$$

(e) Side view of a helix

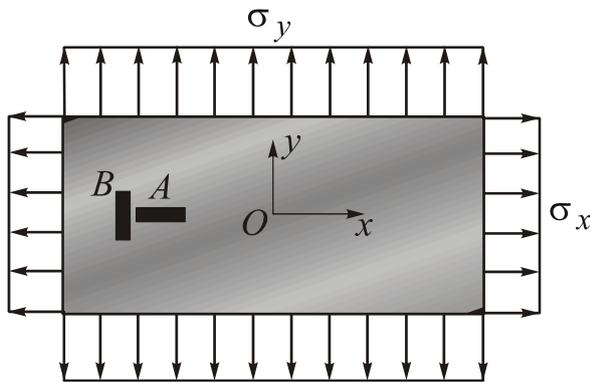
both the pitch and the plate width are established. For practical reasons, the angle  $\theta$  is usually in the range from  $20^\circ$  to  $35^\circ$ .



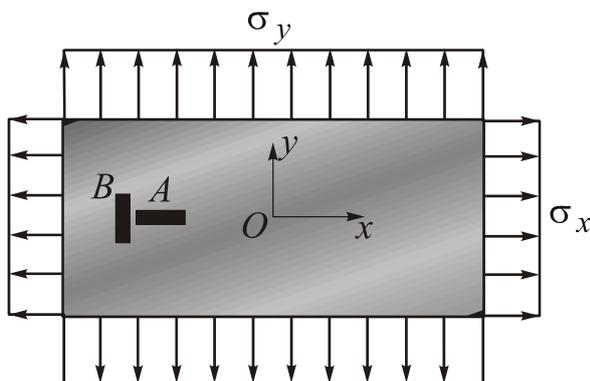
### PROBLEMS

**Problem 3.90** A rectangular and  $y$  directions, respectively, are steel plate with thickness  $t = 6.0 \text{ mm}$  is attached to the plate. The gage readings subjected to uniform normal stresses give normal strains  $\epsilon_x = 0.00062$  and  $\sigma_y$ , as shown in the figure. (elongation) and Strain gages  $A$  and  $B$ , oriented in the  $x$   $\epsilon_y = -0.00045$  (shortening). Knowing

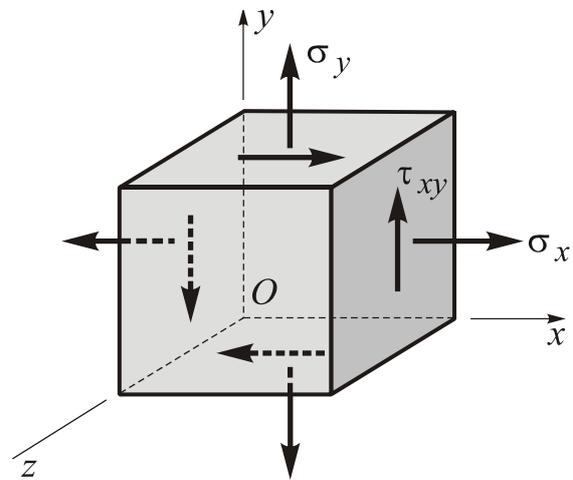
that  $E = 210 \text{ GPa}$  and  $\nu = 0.3$ , determine the stresses  $\sigma_x$  and  $\sigma_y$  and the change  $\Delta t$  in the thickness of the plate.



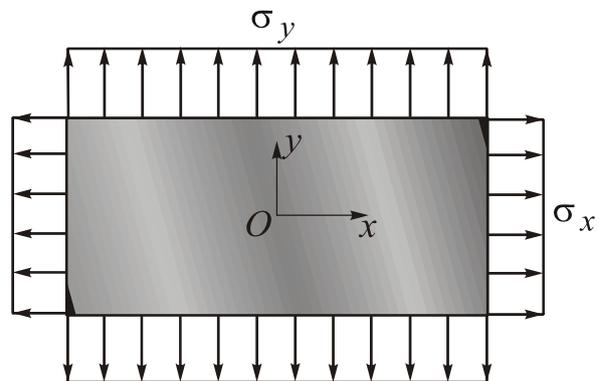
**Problem 3.91** A rectangular aluminum plate with thickness  $t = 8 \text{ mm}$  is subjected to uniform normal stresses  $\sigma_x$  and  $\sigma_y$ , as shown in the figure. Strain gages  $A$  and  $B$ , oriented in the  $x$  and  $y$  directions, respectively, are attached to the plate. The gage readings give normal strains  $\epsilon_x = 300 \times 10^{-6}$  (elongation) and  $\epsilon_y = 80 \times 10^{-6}$  (elongation). Knowing that  $E = 80 \text{ GPa}$  and  $\nu = 0.3$ , determine the stresses  $\sigma_x$  and  $\sigma_y$  and the change  $\Delta t$  in the thickness of the plate.



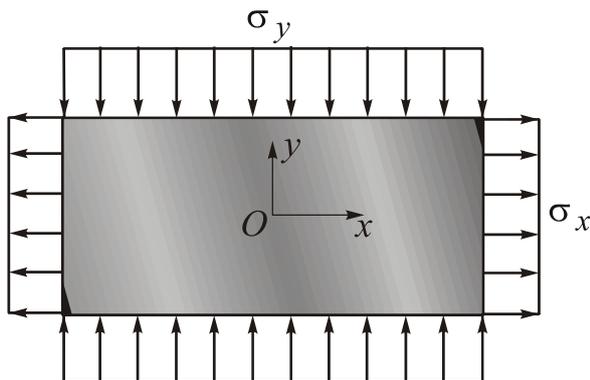
**Problem 3.92** The normal strains  $\epsilon_x$  and  $\epsilon_y$  for an element in plane stress (see figure) are measured with strain gages. (1) Obtain a formula for the normal strain  $\epsilon_z$  in the  $z$  direction in terms of  $\epsilon_x$ ,  $\epsilon_y$ , and Poisson's ratio  $\nu$ . (2) Obtain a formula for the dilatation  $e$  in terms of  $\epsilon_x$ ,  $\epsilon_y$ , and Poisson's ratio  $\nu$ .



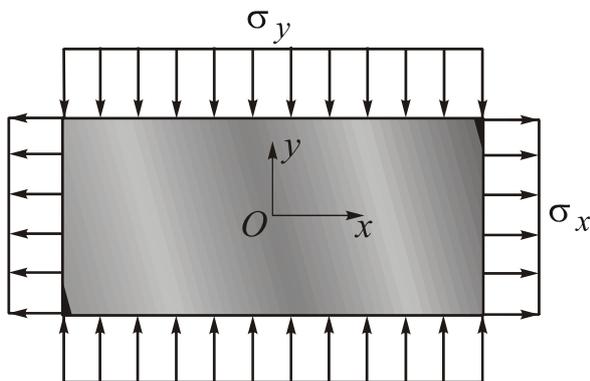
**Problem 3.93** A magnesium plate in biaxial stress is subjected to tensile stresses  $\sigma_x = 30 \text{ MPa}$  and  $\sigma_y = 15 \text{ MPa}$  (see figure). The corresponding strains in the plate are  $\epsilon_x = 550 \times 10^{-6}$  and  $\epsilon_y = 100 \times 10^{-6}$ . Determine Poisson's ratio  $\nu$  and the modulus of elasticity  $E$  for the material.



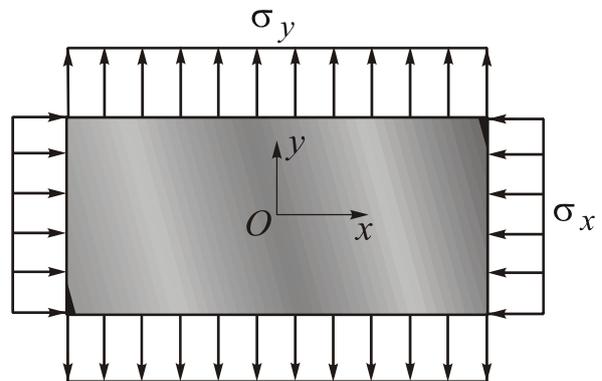
**Problem 3.94** A steel plate in biaxial stress is subjected to stresses  $\sigma_x = 125$  MPa and  $\sigma_y = -62$  MPa (see figure). The corresponding strains in the plate are  $\epsilon_x = 700 \times 10^{-6}$  (elongation) and  $\epsilon_y = -500 \times 10^{-6}$  (shortening). Determine Poisson's ratio  $\nu$  and the modulus of elasticity  $E$  for the material.



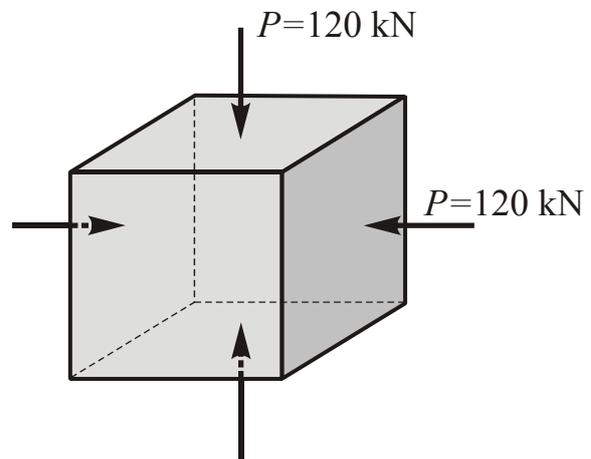
**Problem 3.95** A rectangular plate in biaxial stress (see figure) is subjected to normal stresses  $\sigma_x = 65$  MPa (tension) and  $\sigma_y = -30$  MPa (compression). The plate has dimensions  $200 \times 300 \times 15$  mm and is made of aluminum with  $E = 70$  GPa and  $\nu = 0.33$ . (1) Determine the maximum in-plane shear strain  $\gamma_{\max}$  in the plate. (2) Determine the change  $\Delta t$  in the thickness of the plate. (3) Determine the change  $\Delta V$  in the volume of the plate.



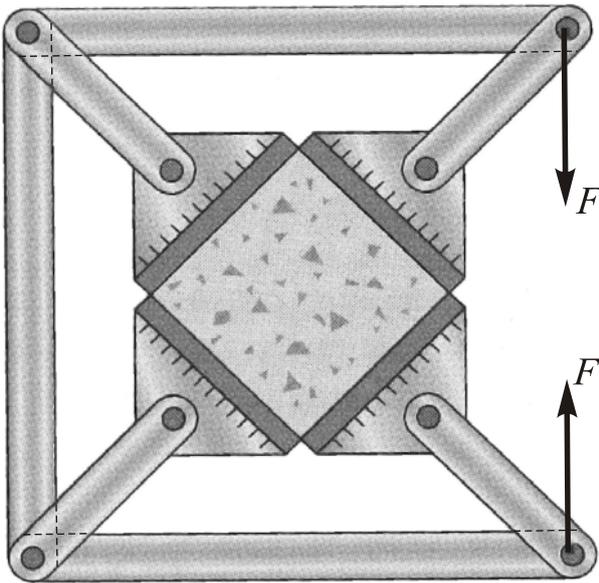
**Problem 3.96** A rectangular plate in biaxial stress (see figure) is subjected to normal stresses  $\sigma_x = -60$  MPa (compression) and  $\sigma_y = 20$  MPa (tension). The plate has dimensions  $200 \times 400 \times 20$  mm and is made of aluminum with  $E = 75$  GPa and  $\nu = 0.30$ . (1) Determine the maximum in-plane shear strain  $\gamma_{\max}$  in the plate. (2) Determine the change  $\Delta t$  in the thickness of the plate. (3) Determine the change  $\Delta V$  in the volume of the plate.



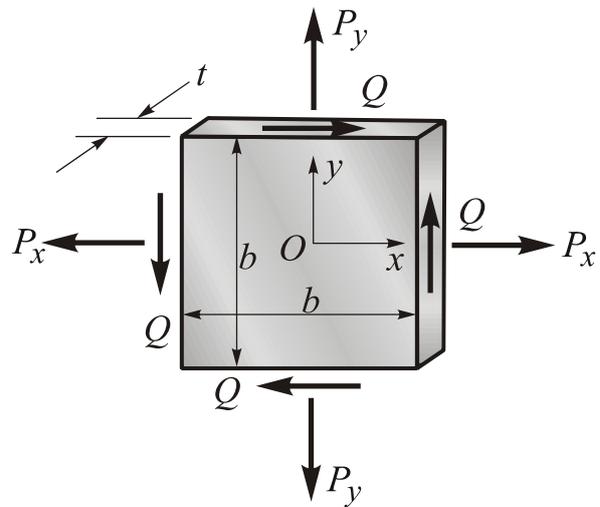
**Problem 3.97** A brass cube 50 mm on each edge is compressed in two perpendicular directions by forces  $P = 100$  kN (see figure). Calculate the decrease  $\Delta V$  in the volume of the cube, assuming  $E = 100$  GPa and  $\nu = 0.34$ .



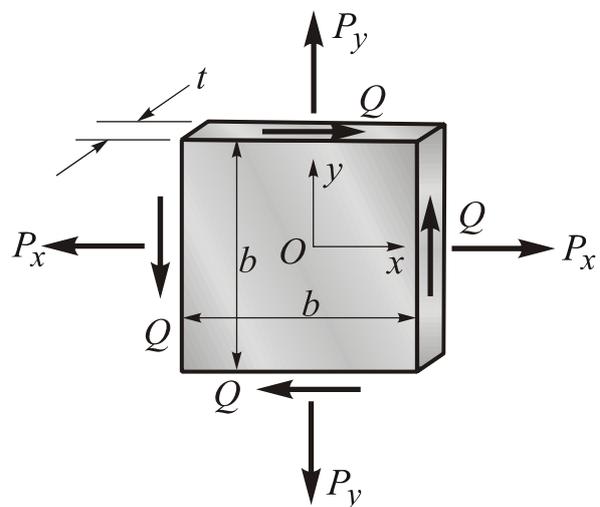
**Problem 3.98** A 100-mm cube of concrete ( $E = 20 \text{ GPa}$ ,  $\nu = 0.1$ ) is compressed in biaxial stress by means of a framework that is loaded as shown in the figure. Assuming that each load  $F$  equals 90 kN, determine the decrease  $\Delta V$  in the volume of the cube.



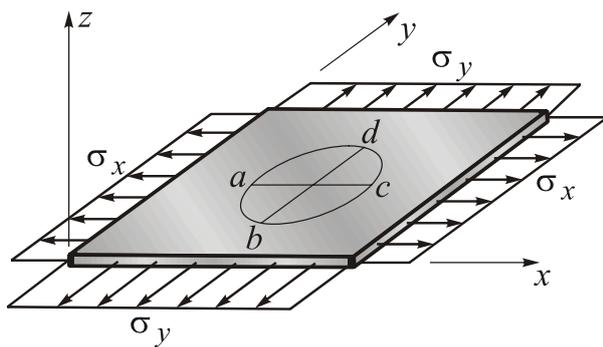
**Problem 3.99** A square plate of width  $b$  and thickness  $t$  is loaded by normal forces  $P_x$  and  $P_y$ , and by shear forces  $Q$ , as shown in the figure. These forces produce uniformly distributed stresses acting on the side faces of the plate. Calculate the change  $\Delta V$  in the volume of the plate if the dimensions are  $b = 500 \text{ mm}$  and  $t = 30 \text{ mm}$ , the plate is made of magnesium with  $E = 45 \text{ GPa}$  and  $\nu = 0.35$ , and the forces are  $P_x = 450 \text{ kN}$ ,  $P_y = 150 \text{ kN}$ , and  $Q = 100 \text{ kN}$ .



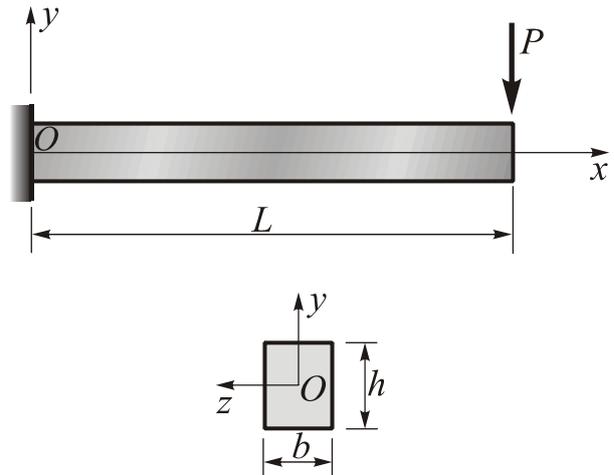
**Problem 3.100** A square steel plate of width  $b$  and thickness  $t$  is loaded by normal forces  $P_x$  and  $P_y$ , and by shear forces  $Q$ , as shown in the figure. These forces produce uniformly distributed stresses acting on the side faces of the plate. Calculate the change  $\Delta V$  in the volume of the plate if the dimensions are  $b = 300 \text{ mm}$  and  $t = 20 \text{ mm}$ , the plate is made of steel with  $E = 210 \text{ GPa}$  and  $\nu = 0.33$ , and the forces are  $P_x = 600 \text{ kN}$ ,  $P_y = 250 \text{ kN}$ , and  $Q = 150 \text{ kN}$ .



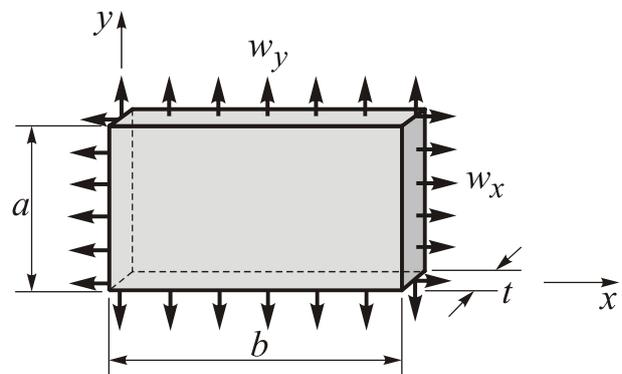
**Problem 3.101** A circle of diameter  $d = 200$  mm is etched on a brass plate (see figure). The plate has dimensions  $500 \times 500 \times 20$  mm. Forces are applied to the plate, producing uniformly distributed normal stresses  $\sigma_x = 40$  MPa and  $\sigma_y = 15$  MPa. Calculate the following quantities: (1) the change in length  $\Delta ac$  of diameter  $ac$ . (2) the change in length  $\Delta bd$  of diameter  $bd$ . (3) the change  $\Delta t$  in the thickness of the plate; (4) the change  $\Delta V$  in the volume of the plate. Assume  $E = 100$  GPa and  $\nu = 0.34$ .



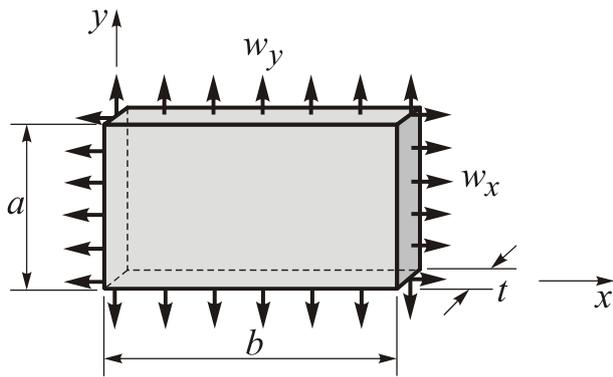
**Problem 3.102** A cantilever beam of length  $L$  and rectangular cross section (width  $b$  and height  $h$ ) supports a concentrated load  $P$  at the free end (see figure). (1) Derive a formula for the increase  $\Delta V_t$  in the volume of the tension half of the beam when the load  $P$  is applied. (2) Derive a formula for the decrease  $\Delta V_c$  in the volume of the compression half of the beam. (3) What is the net change in volume of the entire beam?



**Problem 3.103** A rectangular plate of aluminum ( $E = 70$  GPa,  $\nu = 0.3$ ) is subjected to uniformly distributed loading, as shown in the figure. Determine the values of  $w_x$  and  $w_y$  (in kilonewtons per meter) that produce change in length in the  $x$  direction of 1.5 mm and in the  $y$  direction of 2 mm. Use  $a = 2$  m,  $b = 2$  m, and  $t = 5$  mm.

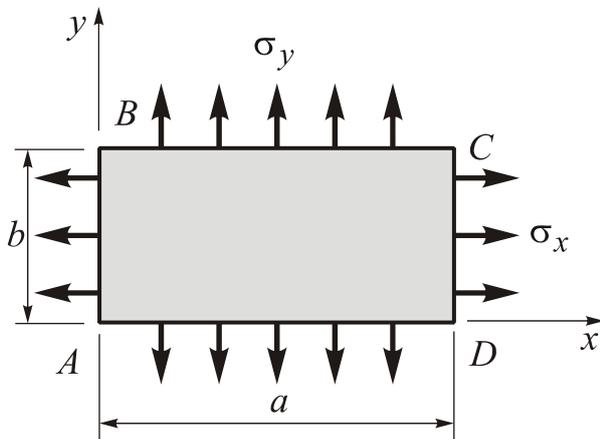
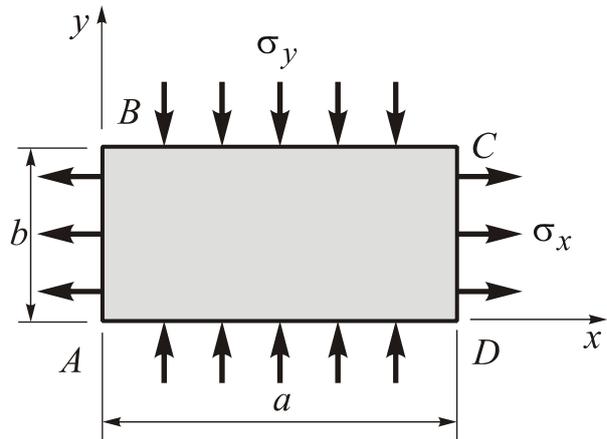


**Problem 3.104** A rectangular steel plate ( $E = 210$  GPa,  $\nu = 0.3$ ) is subjected to uniformly distributed loading, as shown in the figure. Determine the values of  $w_x$  and  $w_y$  (in kilonewtons per meter) that produce change in length in the  $x$  direction of 1.5 mm and in the  $y$  direction of 2 mm. Use  $a = 4$  m,  $b = 2$  m, and  $t = 5$  mm.



**Problem 3.105** A steel plate  $ABCD$  of thickness  $t = 5$  mm is subjected to uniform stresses  $\sigma_x = 120$  MPa and  $\sigma_y = 90$  MPa (see figure). For  $E = 200$  GPa,  $\nu = 0.33$ ,  $a = 160$  mm and  $b = 200$  mm, calculate the change in (1) length of edge  $AB$ ; (2) length of edge  $AD$ ; (3) length of diagonal  $BD$ ; and (4) thickness.

$\sigma_y = -60$  MPa (see figure). For  $E = 200$  GPa,  $\nu = 0.33$ ,  $a = 160$  mm and  $b = 200$  mm, calculate the change in (1) length of edge  $AB$ ; (2) length of edge  $AD$ ; (3) length of diagonal  $BD$ ; and (4) thickness.



**Problem 3.107** A cylindrical steam boiler made of 8-mm steel plate is 1.2 m in inner diameter and 3 m long. Use  $E = 200$  GPa and  $\nu = 0.3$ . For an internal pressure of 3 MPa, calculate (1) the change in the inner diameter; (2) the change in thickness; and (3) the change in length.

**Problem 3.106** A steel plate  $ABCD$  of thickness  $t = 5$  mm is subjected to biaxial loading that results in uniform stresses  $\sigma_x = 100$  MPa and

**Problem 3.108** Verify that the change in radius  $r$  of a sphere subjected to internal pressure  $p$  is given by

$$\Delta r = \frac{pr^2}{2Et}(1 - \nu),$$

where  $t$  – thickness.

### 3.6 Strain-Energy Density and Strain Energy in Plane Stress State

As previously mentioned, the *strain-energy density is the strain energy stored in a unit volume of the material*. For an element in plane stress, we can obtain the strain-energy density by referring to the elements pictured in Figs. 3.23 and 3.24. Because the normal and shear strains occur independently, we can add the strain energies from these two elements to obtain the total energy.

Let us begin by finding the strain energy associated with the normal strains (Fig. 3.23). Since the stress acting on the  $x$ -face of the element is  $\sigma_x$  (see Fig. 3.22), we find that the force acting on the  $x$ -face of the element (Fig. 3.23) is equal to  $\sigma_x bc$ . In structural loading, this force increases gradually from zero to its maximum value. At the same time, the  $x$  face of the element moves through the distance  $a\varepsilon_x$ . Therefore, the work done by this force is

$$\frac{1}{2}(\sigma_x bc)(a\varepsilon_x). \quad (3.71)$$

Similarly, the force  $\sigma_y ac$  acting on the  $y$ -face does work equal to

$$\frac{1}{2}(\sigma_y ac)(b\varepsilon_y). \quad (3.72)$$

The sum of these two terms gives the strain energy stored in the element

$$\frac{abc}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y). \quad (3.73)$$

Thus, the strain-energy density (strain energy per unit volume) due to the normal stresses and strains is

$$U_{01} = \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y). \quad (3.74)$$

The strain-energy density associated with the shear strains (Fig. 3.24) may be evaluated similarly to strain-energy density associated with the normal strains. We begin the analysis by considering a small element of material subjected to shear stresses  $\tau$  on its side faces (Fig. 3.27a). For convenience, we will assume that the front face of the element is square, with each side having length  $h$ . Although the figure shows only a two-dimensional view of the element, we recognize that the element is actually three-dimensional with thickness  $t$  perpendicular to the plane of the figure. Under the action of the shear stresses, the element is distorted so that the front face becomes a rhombus, as shown in Fig. 3.27b. The change in angle at each corner of the element is the shear strain  $\gamma_{xy}$ .

The shear forces  $Q$  acting on the side faces of the element (Fig. 3.27c) are found by multiplying the stresses by the areas  $ht$  over which they act:

$$Q = \tau ht. \quad (3.75)$$

These forces produce work as the element deforms from its initial shape (Fig. 3.27a) to its distorted shape (Fig. 3.27b). To calculate this work we need to determine the relative distances through which the shear forces move. This task is made easier if the element in Fig. 3.27c is rotated as a rigid body until two of its faces are horizontal, as in Fig. 2.27d. During the rigid-body rotation, the net work done by the forces  $Q$  is zero because the forces occur in pairs that form two equal and opposite couples.

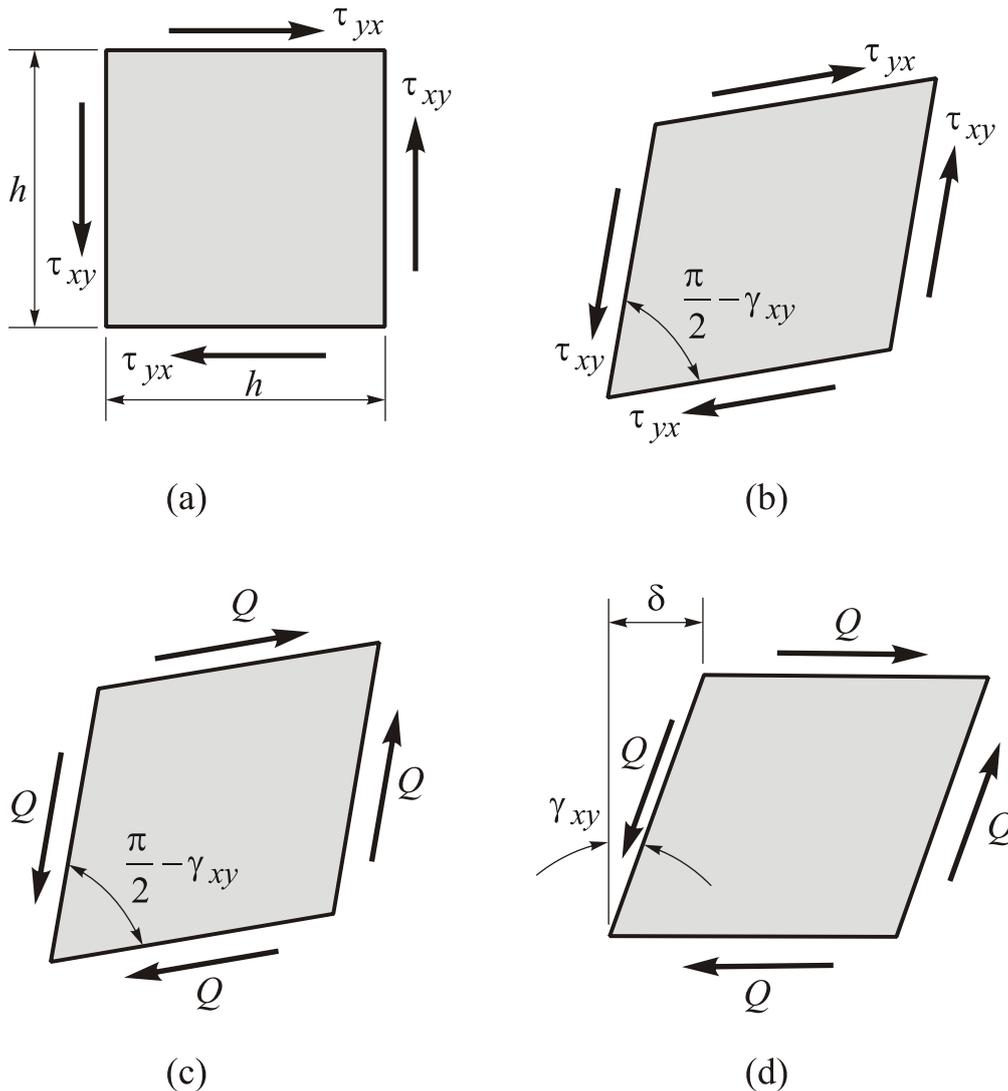


Fig. 3.27 Element in pure shear

As can be seen in Fig. 3.27d, the top face of the element is displaced horizontally through a distance  $\delta$  (relative to the bottom face) as the shear force is gradually increased from zero to its final value  $Q$ . The displacement  $\delta$  is equal to the product of the shear strain  $\gamma_{xy}$  (which is a small angle) and the vertical dimension of the element:

$$\delta = \gamma_{xy}h. \quad (3.76)$$

If we assume that the material is linearly elastic and follows Hooke's law, then the work done by the forces  $Q$  is equal to  $Q\delta/2$ , which is also the strain energy stored in the element:

$$U = W = \frac{Q\delta}{2}. \quad (3.77)$$

Note that the forces acting on the side faces of the element (Fig. 3.27d) do not move along their lines of action – hence they do no work. Substituting from Eqs. (3.75) and (3.76) into Eq. (3.77), we get the total strain energy of the element:

$$U = \frac{\tau\gamma_{xy}h^2t}{2}. \quad (3.78)$$

Because the volume of the element is  $h^2t$ , the strain-energy density in pure shear  $U_{0_2}$  is

$$U_{0_2} = \frac{\tau_{xy}\gamma_{xy}}{2}. \quad (3.79)$$

By combining the strain-energy densities for the normal and shear strains, we obtain the following formula for the strain-energy density in plane stress:

$$U_0 = \frac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy}). \quad (3.80)$$

Substituting for the strains from Eqs. (3.47) through (3.50), we obtain the strain-energy density in terms of stresses:

$$U_0 = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y) + \frac{\tau_{xy}^2}{2G}. \quad (3.81)$$

In a similar manner, we can substitute for the stresses from Eqs. (3.51) through (3.53) and obtain the strain-energy density in terms of strains:

$$U_0 = \frac{E}{2(1-\nu^2)}(\varepsilon_x^2 + \varepsilon_y^2 - 2\nu\varepsilon_x\varepsilon_y) + \frac{G\gamma_{xy}^2}{2}. \quad (3.82)$$

To obtain the strain-energy density in the special case of *biaxial stress*, we simply drop the shear terms in Eqs. (3.80), (3.81), and (3.82).

For the special case of *uniaxial stress*, we substitute the following values

$$\sigma_y = 0, \quad \tau_{xy} = 0, \quad \varepsilon_y = -\nu\varepsilon_x, \quad \gamma_{xy} = 0$$

into Eqs. (3.81) and (3.82) and obtain, respectively,

$$U_0 = \frac{\sigma_x^2}{2E}, \quad \text{or} \quad U_0 = \frac{E\varepsilon_x^2}{2}. \quad (3.83)$$

These equations agree with Eq. (2.21) in Section 2.3. Also, for *pure shear* we substitute

$$\sigma_x = \sigma_y = 0, \quad \varepsilon_x = \varepsilon_y = 0$$

into Eqs. (3.77) and (3.78) and obtain

$$U_0 = \frac{\tau_{xy}^2}{2G}, \quad \text{or} \quad U_0 = \frac{G\gamma_{xy}^2}{2}. \quad (3.84)$$

These equations are similar in form to those for uniaxial stress (Eq. 3.79) and agree with Eq. (2.24) of Section 2.3.

The strain energy for plane stress state may be expressed as the integral

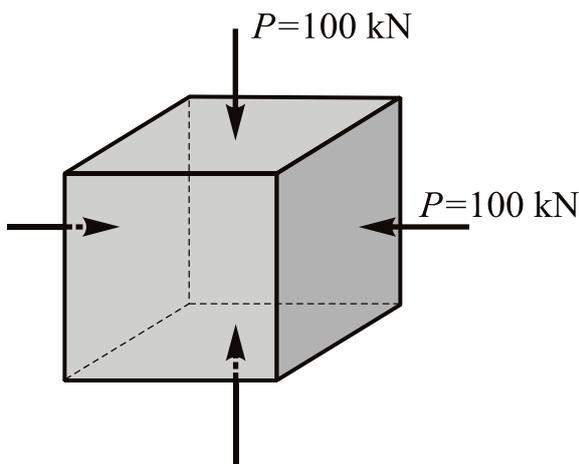
$$U = \int_V U_0 dV = \frac{1}{2} \int_V \left[ \frac{1}{E} (\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y) + \frac{\tau_{xy}^2}{G} \right] dV. \quad (3.85)$$

The integration is produced over the volume of the member. **Note, that the strain energy is a nonlinear (quadratic) function of loading or deformation. The principle of superposition is therefore not valid for strain energy.**



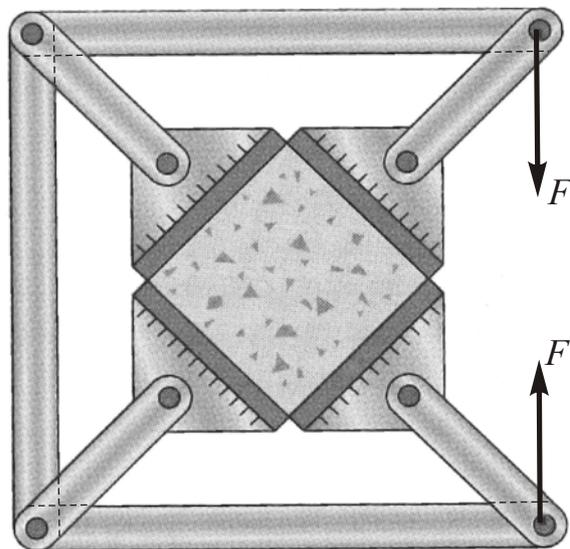
## PROBLEMS

**Problem 3.110** A brass cube 20 mm on each edge is compressed in two perpendicular directions by forces  $P=100$  kN (see figure). Calculate the strain energy  $U$  stored in the cube, assuming  $E=90$  GPa and  $\nu=0.34$ .



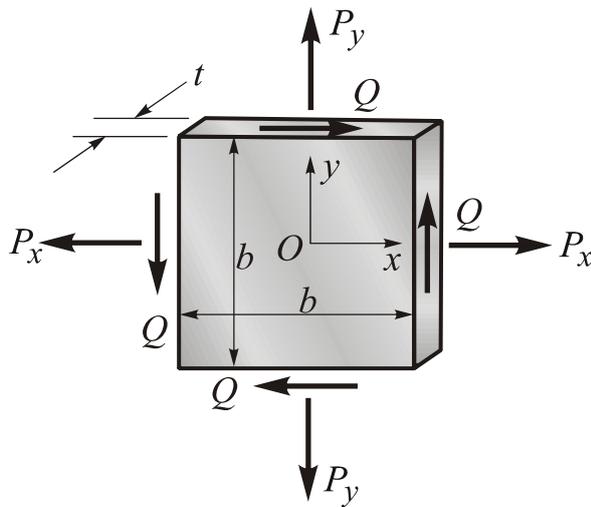
**Problem 3.111** A 7-cm cube of concrete ( $E=20$  GPa,  $\nu=0.1$ ) is compressed in biaxial stress by means of a framework that is loaded as shown

in the figure. Assuming that each load  $F$  equals 50 kN, determine the strain energy  $U$  stored in the cube.

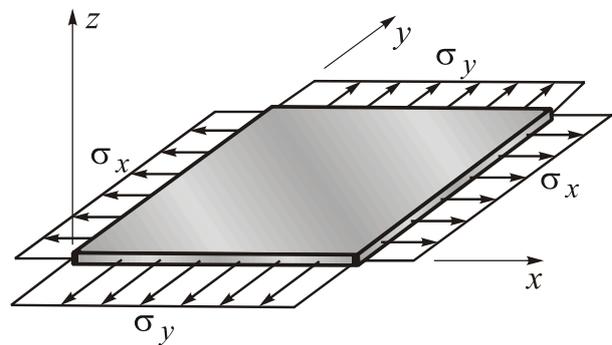


**Problem 3.112** A square plate of width  $b$  and thickness  $t$  is loaded by normal forces  $P_x$  and  $P_y$ , and by shear forces  $Q$ , as shown in the figure. These forces produce uniformly distributed

stresses acting on the side faces of the plate. Calculate the strain energy  $U$  stored in the plate if the dimensions are  $b = 100$  mm and  $t = 10$  mm, the plate is made of magnesium with  $E = 45$  GPa and  $\nu = 0.35$ , and the forces are  $P_x = 150$  kN,  $P_y = 150$  kN, and  $Q = 80$  kN.



**Problem 3.113** Brass plate (see figure) has dimensions  $400 \times 400 \times 10$  mm. Forces are applied to the plate, producing uniformly distributed normal stresses  $\sigma_x = 30$  MPa and  $\sigma_y = 20$  MPa. Calculate the strain energy  $U$  stored in the plate. Assume  $E = 100$  GPa and  $\nu = 0.34$ .



### 3.7 Variation of Stress Throughout Deformable Solid. Differential Equations of Equilibrium

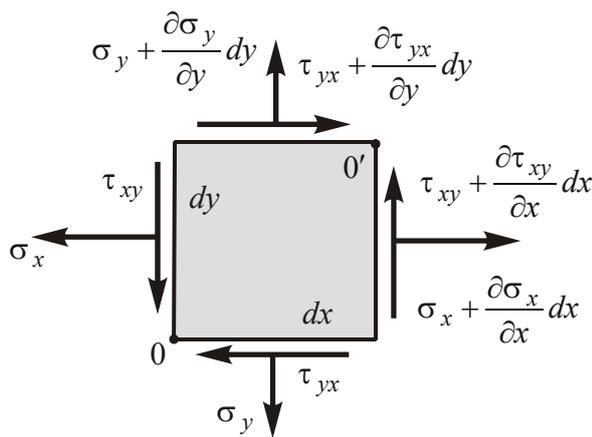


Fig. 3.28 Stress variation on an element

thickness are shown in Fig. 3.28. As we move from point  $O$  to  $O'$ , the increment of stress may be expressed by a truncated *Taylor's expansion*:  $\sigma_x + (\partial \sigma_x / \partial x) dx$ .

The components of stress state generally vary from point to point in a loaded deformable body. Such variations of stress, accounted for by the theory of elasticity, are governed by the equations of statics. Fulfillment of these requirements establishes certain relationships referred to as the *differential equations of equilibrium*.

For a two-dimensional case, the stresses acting on an infinitesimally element of sides  $dx$ ,  $dy$  and of unit

The partial derivative is used as  $\sigma_x$  varies with  $x$  and  $y$ . The stresses  $\sigma_y$ ,  $\tau_{xy}$ , and  $\tau_{yx}$  similarly change.

Evidently, that the element of Fig. 3.28 must satisfy the condition  $\sum M_0 = 0$ :

$$\begin{aligned} \left( \frac{\partial \sigma_y}{\partial y} dx dy \right) \frac{dx}{2} - \left( \frac{\partial \sigma_x}{\partial x} dx dy \right) \frac{dy}{2} + \left( \tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx \right) dx dy - \\ - \left( \tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy \right) dx dy = 0. \end{aligned} \quad (3.86)$$

Neglecting the triple products involving  $dx$  and  $dy$ , we have

$$\tau_{xy} = \tau_{yx},$$

as already obtained in Chapter 2.

The equilibrium of  $x$ -directed forces,  $\sum F_x = 0$ , yields

$$\left( \sigma_x + \frac{\partial \sigma_x}{\partial x} dx \right) dy - \sigma_x dy + \left( \tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} dy \right) dx - \tau_{xy} dx = 0. \quad (3.87)$$

A similar expression is written for  $\sum F_y = 0$ . Simplifying these relationships, we have

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0. \end{aligned} \quad (3.88)$$

This differential equations of equilibrium apply for any type of material. These relationships show that the rate of change of normal stress must be accompanied by a rate of change in shearing stress. As Eqs. (3.88) contain the three unknown stresses  $\sigma_x$ ,  $\sigma_y$ ,  $\tau_{xy}$ , *problems in stress analysis are internally statically indeterminate*. In the mechanics of materials, this *indeterminacy is eliminated by introducing appropriate assumptions and considering the equilibrium of finite segments of a member*. The equations for the case of three-dimensional stress may be generalized from the foregoing expressions by referring to equations of statics:

$$\sum F_x = 0, \sum F_y = 0, \sum F_z = 0, \sum M_x = 0, \sum M_y = 0, \sum M_z = 0. \quad (3.85)$$

The complete analysis of structural members by the *method of equilibrium* requires consideration of a number of conditions relating to certain *laws of forces*,

*laws of material deformation, and geometrical conditions.* This essential relationships, referred to as *basic principles of analysis* are:

*Statics.* The equations of equilibrium of forces must be satisfied through the member.

*Deformations.* The stress-strain or force-deformation relations (for example, Hooke's law) must apply to the behavior of the material on which the member is constructed.

*Geometry.* The conditions of geometric fit or *compatibility of deformations* must be satisfied; that is, each deformed portion of the member must fit together with adjacent portions.

In the case of a two-dimensional problem in elasticity, it is required that the following eight quantities be ascertained:  $\sigma_x, \sigma_y, \tau_{xy}, \varepsilon_x, \varepsilon_y, \gamma_{xy}, u$  and  $v$ . Here  $u, v$  – components of the displacement of the point. These components must satisfy eight governing equations throughout the member in addition to the boundary conditions:

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x}, & \varepsilon_y &= \frac{\partial v}{\partial y}, & \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \\ \varepsilon_x &= \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E}, & \varepsilon_y &= \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E}, & \gamma_{xy} &= \frac{\tau_{xy}}{G}, \\ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0, & \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} &= 0. \end{aligned} \quad (3.89)$$

Solutions applying the methods of elasticity are not presented in this text.

## Chapter 4 Triaxial Stress

An element of deformable solid subjected to only normal stresses  $\sigma_x$ ,  $\sigma_y$ , and  $\sigma_z$  acting in three mutually perpendicular directions is said to be in a *state of triaxial stress* (Fig. 4.1a). Since there are no shear stresses on the  $x$ ,  $y$ , and  $z$  faces, the stresses  $\sigma_x$ ,  $\sigma_y$ , and  $\sigma_z$  are *the principal stresses* in the material (*remember that they are named as  $\sigma_1$ ,  $\sigma_2$ ,  $\sigma_3$ , while  $\sigma_1 \geq \sigma_2 \geq \sigma_3$* ).

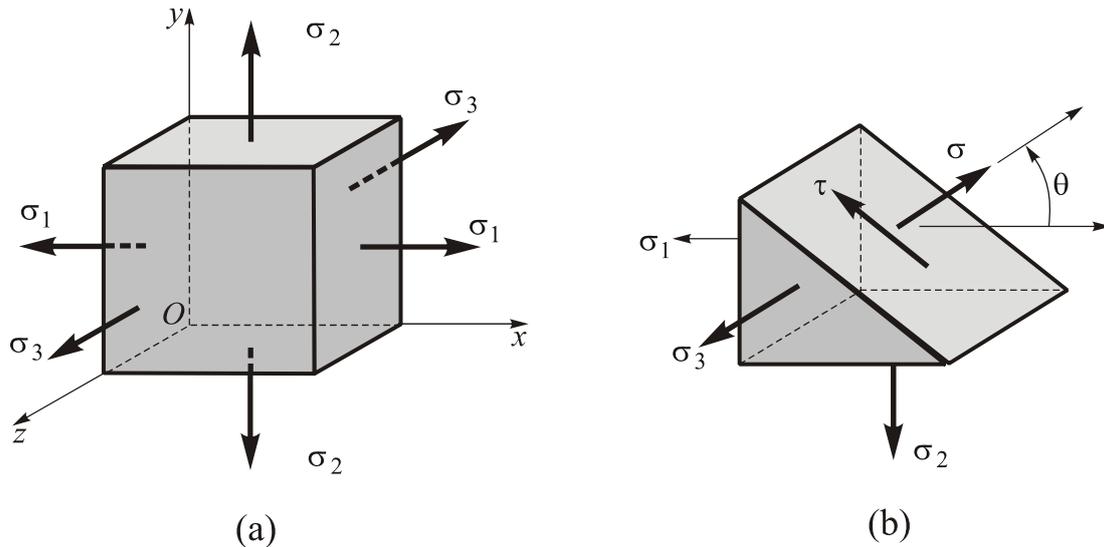


Fig. 4.1 Element in triaxial stress

In the case, when an inclined plane parallel to the  $z$  axis is cut through the element (Fig. 4.1b), the only *stresses on the inclined face are the normal stress  $\sigma$  and shear stress  $\tau$* , both of which act parallel to the  $xy$  plane. Because the *stresses  $\sigma$  and  $\tau$*  (Fig. 4.1b) are found from equations of force equilibrium in the  $xy$  plane, they are *independent of the normal stress  $\sigma_z$* . Therefore, we can use the transformation equations of plane stress, as well as Mohr's circle for plane stress, when determining the stresses  $\sigma$  and  $\tau$ . The same general conclusion holds for the normal and shear stresses acting on inclined planes cut through the element parallel to the  $x$  and  $y$  axes.

### 4.1 Maximum Shear Stresses

We know that in plane stress state, the maximum shear stresses occur on planes oriented at  $45^\circ$  to the principal planes. Therefore, for a material in triaxial stress (Fig. 4.1a), the maximum shear stresses occur on elements oriented at angles of  $45^\circ$  to the  $x$ ,  $y$ , and  $z$  axes. For example, consider an element obtained by a  $45^\circ$  rotation about the  $z$  axis. The maximum positive and negative shear stresses acting on this element are

$$(\tau_{\max})_z = \pm \frac{\sigma_x - \sigma_y}{2} = \pm \frac{\sigma_1 - \sigma_2}{2}. \quad (4.1)$$

Similarly, by rotating about the  $x$  and  $y$  axes through angles of  $45^\circ$ , we obtain the following maximum shear stresses:

$$(\tau_{\max})_x = \pm \frac{\sigma_y - \sigma_z}{2} = \pm \frac{\sigma_2 - \sigma_3}{2}, \tag{4.2}$$

$$(\tau_{\max})_y = \pm \frac{\sigma_x - \sigma_z}{2} = \pm \frac{\sigma_1 - \sigma_3}{2}. \tag{4.3}$$

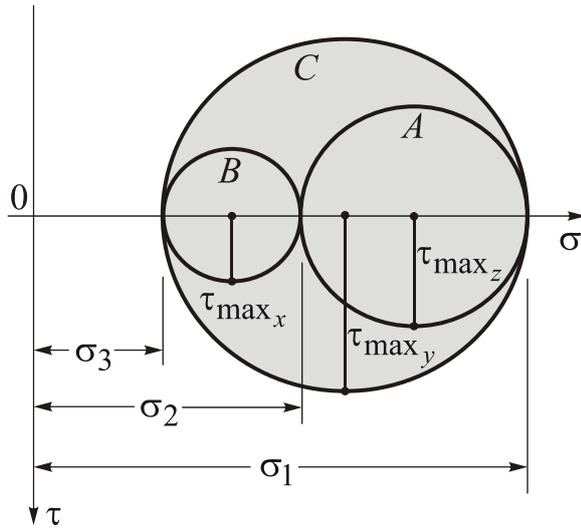


Fig. 4.2 Mohr's circles for an element in triaxial stress

The stresses acting on elements oriented at various angles to the  $x$ ,  $y$ , and  $z$  axes can be visualized with the aid of Mohr's circles. For elements oriented by rotations about the  $z$  axis, the corresponding circle is labeled  $A$  in Fig. 4.2. **Note that this circle is drawn for the case in  $\sigma_1 > \sigma_2$  and both  $\sigma_1$  and  $\sigma_2$  are tensile stresses.**

stresses given by Eqs. (4.1, 4.2, and 4.3), and the absolute *maximum shear stress is equal to the radius of the largest circle. The normal stresses acting on the planes of maximum shear stresses have magnitudes given by the abscissas of the centers of the respective circles.*

In a similar manner, we can construct circles and for elements oriented by rotations about the  $x$  and  $y$  axes, respectively. The radii of the circles represent the maximum shear stresses

Considered stresses act on planes obtained by rotating about the  $x$ ,  $y$ , and  $z$  axes. We can also cut through the element in skew directions, so that the resulting inclined planes are skew to all three coordinate axes. The normal and shear stresses acting on such planes can be obtained by a more complicated three-dimensional analysis. However, the *normal stresses acting on skew planes are intermediate in value between the algebraically maximum and minimum principal stresses, and the shear stresses on those planes are smaller (in absolute value) than the absolute maximum shear stress obtained from Eqs. (4.1, 4.2, and 4.3).*

### 4.2 Hooke's Law for Triaxial Stress. Generalized Hooke's Law

The uniaxial stress-strain relation (see Chapter 2) may be extended to include biaxial and triaxial states of stress often encountered in engineering applications. In the derivations which follow, we rely upon certain experimental

evidence: a normal stress causes no shearing strain whatever, and a shearing stress produces only a shearing strain. Also, the principle of superposition applies under multiaxial loading, since the strain components are small quantities. These assumptions are limited to isotropic materials stressed in the linearly elastic range.

Consider one more a structural element of unit thickness subjected to a biaxial state of stress (Fig. 4.3a). Under the action of the stress  $\sigma_x$ , not only would the direct strain  $\sigma_x/E$  occur, but a  $y$  contraction as well,  $-\nu\sigma_x/E$ , indicated by the dashed lines in the figure. Similarly, were  $\sigma_y$ , to act alone, an  $x$  contraction  $-\nu\sigma_y/E$  and a  $y$  strain  $\sigma_y/E$  would result. Thus simultaneous action of both stresses  $\sigma_x$  and  $\sigma_y$  results in the following strains in the  $x$  and  $y$  directions:

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E}, \quad (4.4)$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E}. \quad (4.5)$$

The elastic stress-strain relation, under the state of *pure shear* (Fig. 4.3b), is of the form

$$\gamma_{xy} = \frac{\tau_{xy}}{G}. \quad (4.6)$$

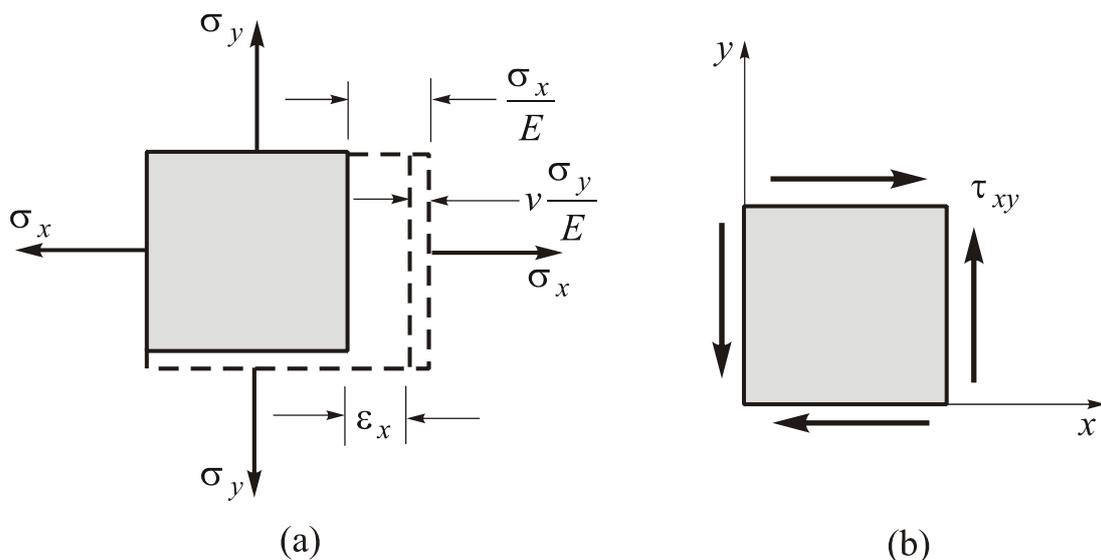


Fig. 4.3 Element in state of (a) biaxial stress and (b) pure shear.

That is,  $\tau_{xy}$  produces only its corresponding shearing strain  $\gamma_{xy}$ . From Eqs. (4.4., 4.5, and 4.6) we obtain the following stress-strain relations mentioned above as Eqs. (3.51) through (3.53):

$$\sigma_x = \frac{E}{1-\nu^2} (\varepsilon_x + \nu\varepsilon_y), \quad (4.7)$$

$$\sigma_y = \frac{E}{1-\nu^2} (\varepsilon_y + \nu\varepsilon_x), \quad (4.8)$$

$$\tau_{xy} = G\gamma_{xy}. \quad (4.9)$$

Equations (4.4 – 4.6) through (4.7 – 4.9) represent Hooke's law for two-dimensional stress.

An identical analysis enables one to connect the components  $\varepsilon_z, \gamma_{yz}, \gamma_{xz}$  of strain with stress and material properties. The foregoing procedure is readily

extended to a *three-dimensional state of stress* (see Fig. 4.4). Then the strain-stress relations, known as the *generalized Hooke's law*, consist of the expressions:

$$\varepsilon_x = \frac{1}{E} [\sigma_x - \nu(\sigma_y + \sigma_z)], \quad (4.10)$$

$$\varepsilon_y = \frac{1}{E} [\sigma_y - \nu(\sigma_x + \sigma_z)], \quad (4.11)$$

$$\varepsilon_z = \frac{1}{E} [\sigma_z - \nu(\sigma_x + \sigma_y)], \quad (4.12)$$

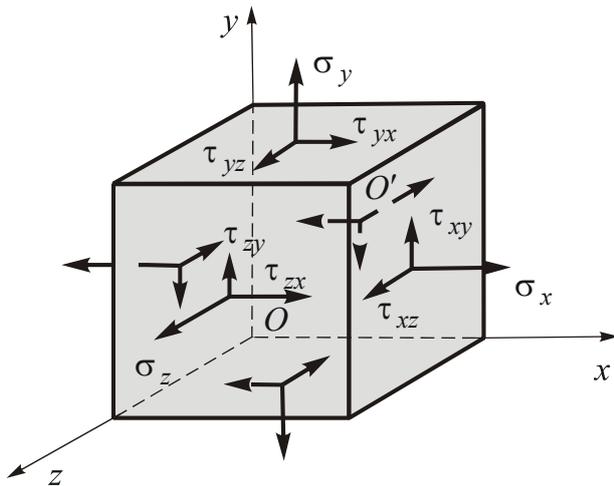


Fig. 4.4 Three-dimensional state of stress

$$\gamma_{xy} = \frac{\tau_{xy}}{G}, \quad (4.13) \quad \gamma_{yz} = \frac{\tau_{yz}}{G}, \quad (4.14) \quad \gamma_{xz} = \frac{\tau_{xz}}{G}. \quad (4.15)$$

Positive value for a stress (strain) component signifies tension (extension), and a negative value compression (contraction). **Note, if a particular normal stress is compressive, the sign of the corresponding term in Eqs. (4.10 – 4.15) changes.**

*In triaxial stress, when shear stresses are absent, and normal stresses became principal, Hooke's law has more simple configuration.* Equations for the strains:

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E} (\sigma_y + \sigma_z), \quad \text{or} \quad \varepsilon_1 = \frac{\sigma_1}{E} - \frac{\nu}{E} (\sigma_2 + \sigma_3), \quad (4.16)$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_z + \sigma_x), \quad \text{or} \quad \varepsilon_2 = \frac{\sigma_2}{E} - \frac{\nu}{E}(\sigma_1 + \sigma_3), \quad (4.17)$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_x + \sigma_y), \quad \text{or} \quad \varepsilon_3 = \frac{\sigma_3}{E} - \frac{\nu}{E}(\sigma_1 + \sigma_2). \quad (4.18)$$

In these equations, the standard sign conventions are used; that is, tensile stress  $\sigma$  and extensional strain  $\varepsilon$  are positive.

The preceding equations can be solved simultaneously for the stresses in terms of the strains:

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)} \left[ (1-\nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z) \right], \quad (4.19)$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)} \left[ (1-\nu)\varepsilon_y + \nu(\varepsilon_z + \varepsilon_x) \right], \quad (4.20)$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)} \left[ (1-\nu)\varepsilon_z + \nu(\varepsilon_x + \varepsilon_y) \right]. \quad (4.21)$$

In the special case of *biaxial stress*, we can obtain the equations of Hooke's law by substituting  $\sigma_z = 0$  into the above Eqs. (4.16 – 4.18).

### 4.3 Unit Volume Change

The unit volume change (or dilatation) for an element in triaxial stress is obtained in the same manner as for plane stress (see Section 3.5.3). If the element is subjected to strains  $\varepsilon_x$ ,  $\varepsilon_y$ , and  $\varepsilon_z$ , we may use Eq. (3.66) for the unit volume change:

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z. \quad (4.22)$$

**Note that this equation is valid for any material provided the strains are small.**

If Hooke's law holds for the material, we can substitute for the strains  $\varepsilon_x$ ,  $\varepsilon_y$ , and  $\varepsilon_z$  from Eqs. (4.16 through 4.18) and obtain

$$e = \frac{1-2\nu}{E} (\sigma_x + \sigma_y + \sigma_z). \quad (4.23)$$

Equations (4.22) and (4.23) give the *unit volume change in triaxial stress* in terms of the principal strains and stresses, respectively.

#### 4.4 Strain-Energy Density in Triaxial and Three-Dimensional Stress

The strain-energy density for an element in triaxial stress is obtained by the same method used for plane stress (see Section 3.6). When principal stresses  $\sigma_x$  and  $\sigma_y$  act alone (biaxial stress), the strain-energy density is

$$U_0 = \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y). \quad (4.24)$$

When the element is in triaxial stress and subjected to principal stresses  $\sigma_x$ ,  $\sigma_y$ , and  $\sigma_z$  the expression for strain-energy density becomes

$$U_0 = \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z). \quad (4.25)$$

Substituting for the strains from Eqs. (4.16) through (4.18), we obtain the strain-energy density in terms of the principal stresses:

$$U_0 = \frac{1}{2}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\nu}{E}(\sigma_x \sigma_y + \sigma_x \sigma_z + \sigma_y \sigma_z). \quad (4.26)$$

In a similar manner, but using Eqs. (4.19) through (4.21), we can express the strain-energy density in terms of the principal strains:

$$U_0 = \frac{E}{2(1+\nu)(1-2\nu)} \left[ (1-\nu)(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2) + 2\nu(\varepsilon_x \varepsilon_y + \varepsilon_x \varepsilon_z + \varepsilon_y \varepsilon_z) \right]. \quad (4.27)$$

For general (three-dimensional) state of stress it will be

$$U_0 = \frac{1}{2E} \left[ \sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\nu(\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x) \right] + \frac{1}{2G} (\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2). \quad (4.28)$$

**Note.** When calculating from these expressions, we must be sure to substitute the stresses and strains with their proper algebraic signs.

The total strain energy stored in an elastic body can be obtained by integrating the strain-energy density over the entire volume:

$$U = \iiint U_0 dx dy dz = \int U_0 dV. \quad (4.29)$$

Using this expression, we can evaluate the strain energy for members under combined loading. **Note, that the strain energy is a nonlinear (quadratic) function of loading or deformation.** *The principle of superposition is therefore not valid for strain energy.*

### 4.5 Spherical Stress

A special type of triaxial stress, called *spherical stress*, occurs whenever all three normal stresses are equal (Fig. 4.5):

$$\sigma_x = \sigma_y = \sigma_z = \sigma_0. \quad (4.30)$$

Under these stress conditions, *any* plane cut through the element will be subjected to the same normal stress  $\sigma_0$  and will be free of shear stress. Thus, we have equal normal stresses in every direction and no shear stresses. *Every plane is a principal plane*, and the three Mohr's circles shown in Fig. 4.2 reduce to a single point.

The normal strains in spherical stress are also the same in all directions, provided the material is homogeneous and isotropic. If Hooke's law applies, the normal strains are as obtained from Eqs. (4.16) through (4.18).

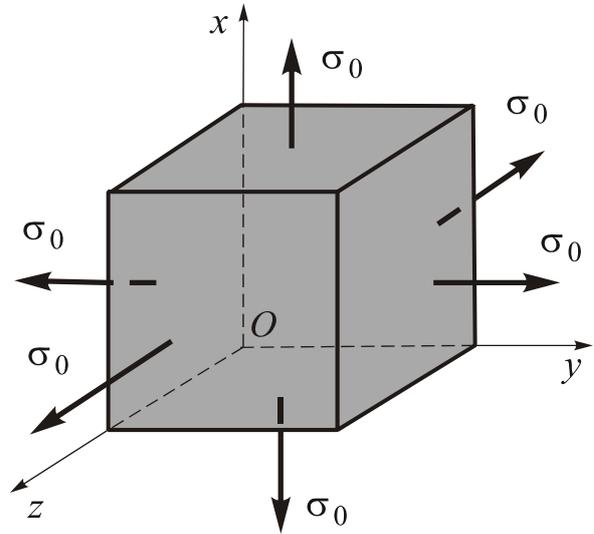


Fig. 4.5 Element in spherical stress

$$\varepsilon_0 = \frac{\sigma_0}{E}(1 - 2\nu), \quad (4.31)$$

Since there are no shear strains, an *element in the shape of a cube changes in size but remains a cube*. In general, any solid subjected to spherical stress will maintain its relative proportions but will expand or contract in volume depending upon whether  $\sigma_0$  is tensile or compressive.

The expression for the *unit volume change* can be obtained from Eq. (4.22) by substituting for the strains from Eq. (4.29). The result is

$$e = 3\varepsilon_0 = \frac{3\sigma_0(1 - 2\nu)}{E}. \quad (4.32)$$

Equation (4.32) is usually expressed in more compact form by introducing a new quantity called the *volume modulus of elasticity*, or *bulk modulus of elasticity*, which is defined as follows:

$$K = \frac{E}{3(1 - 2\nu)}. \quad (4.33)$$

With this notation, the expression for the unit volume change becomes

$$e = \frac{\sigma_0}{K}, \quad (4.34)$$

and the volume modulus is

$$K = \frac{\sigma_0}{e}. \quad (4.35)$$

Thus, the volume modulus can be defined as the ratio of the spherical stress to the volumetric strain, which is analogous to the definition of the modulus  $E$  in uniaxial stress. **Note, that the preceding formulas for  $e$  and  $K$  are based upon the assumption that the strains are small and that Hooke's law holds for the material.**

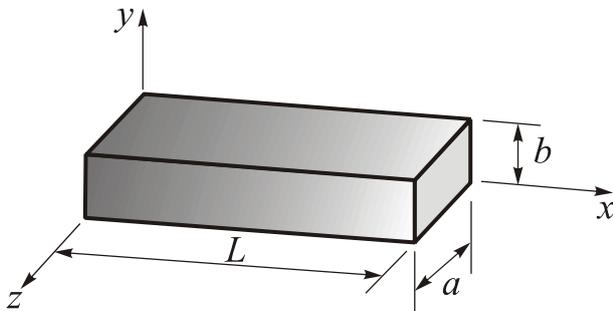
From Eq. (4.33) for  $K$ , we see that if Poisson's ratio  $\nu$  equals 0.33, the moduli  $K$  and  $E$  are numerically equal. If  $\nu = 0$ , then  $K$  has the value  $E/3$ , and if  $\nu = 0.5$ ,  $K$  becomes infinite, which corresponds to an absolutely rigid material having no change in volume (that is, the *material is incompressible*). Thus, the theoretical maximum value of Poisson's ratio is 0.5.

The formulas for spherical stress were derived for an element subjected to uniform tension in all directions, but of course the formulas also apply to an element in uniform compression. In the case of uniform compression, the stresses and strains have negative signs. Uniform compression occurs when the material is subjected to a pressure  $p$ . This state of stress is often called *hydrostatic stress*.



## EXAMPLES

### Example 4.1



A steel rectangular block is subjected to a uniform pressure of  $p = 150$  MPa acting on all faces. Calculate the change in volume and dimensions for  $a = 40$  mm,  $b = 30$  mm, and  $L = 100$  mm. Use  $E = 200$  GPa and  $\nu = 0.3$ .

**Solution** Inserting  $\sigma_x = \sigma_y = \sigma_z = \sigma_0 = -p$  into Eqs. (4.10) through (4.15) and setting  $\varepsilon_x = \varepsilon_y = \varepsilon_z = \varepsilon$  we obtain

$$\varepsilon_0 = -\frac{p}{E}(1 - 2\nu).$$

The given numerical values are substituted into the above to yield

$$\varepsilon_0 = -\frac{150 \times 10^6}{200 \times 10^9}(1 - 0.6) = -0.3 \times 10^{-3}.$$

Thus the change in the volume of the block, using Eqs. (3.66) or (4.22), is  $\Delta V = -3\varepsilon_0(abL) = -108 \text{ mm}^3$ . Deformations in the  $x, y$ , and  $z$  directions are, respectively,

$$\Delta l_x = \delta_x = (-300 \times 10^{-6})(100) = -0.03 \text{ mm},$$

$$\Delta l_y = \delta_y = (-300 \times 10^{-6})(30) = -0.009 \text{ mm},$$

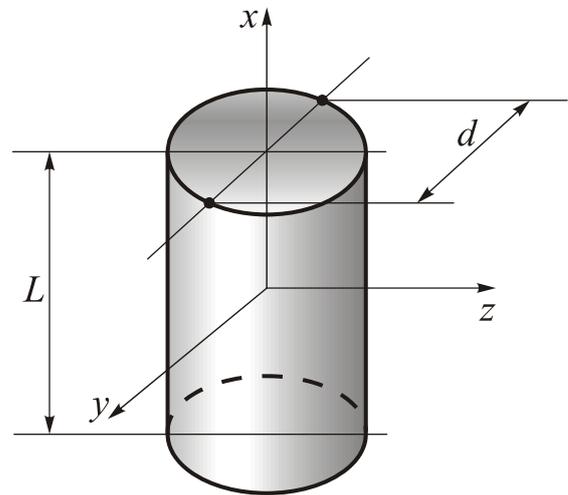
$$\Delta l_z = \delta_z = (-300 \times 10^{-6})(40) = -0.012 \text{ mm},$$

where the minus sign indicates contraction.

### Example 4.2

A solid cast-iron cylinder (see figure) is subjected to axial and radial compressive stresses 40 and 10 MPa, respectively. For  $E = 100 \text{ GPa}$ ,  $\nu = 0.25$ ,  $d = 120 \text{ mm}$ , and  $L = 200 \text{ mm}$ , determine the change in (1) the length  $\Delta L$  and diameter  $\Delta d$  and (2) the volume of the cylinder  $\Delta V$ .

**Solution** Note that  $\sigma_x = -40 \text{ MPa}$  and along any radius  $\sigma_y = \sigma_z = \sigma_0 = -10 \text{ MPa}$ . The



corresponding axial and radial strains, using Eqs. (4.10 through 4.15), are

$$\varepsilon_x = -\frac{1}{E}[\sigma_x - \nu(\sigma_0 + \sigma_0)] = -\frac{10^6}{100 \times 10^9} \left[ 40 - \frac{1}{4}(10 + 10) \right] = -350 \times 10^{-6},$$

$$\varepsilon_y = \varepsilon_z = \varepsilon = -\frac{1}{E}[\sigma_0 - \nu(\sigma_0 + \sigma_x)] = -\frac{1}{100 \times 10^9} \left[ 10 - \frac{1}{4}(10 + 40) \right] = 25 \times 10^{-6}.$$

(1) The decrease in length and increase in diameter are

$$\Delta L = \varepsilon_x L = (-350 \times 10^{-6})(200) = -0.07 \text{ mm},$$

$$\Delta d = \varepsilon d = (25 \times 10^{-6})(120) = 0.003 \text{ mm}.$$

(2) The decrease in volume is determined from Eq. (4.22),

$$e = \varepsilon_x + 2\varepsilon_0 = (-350 + 2 \times 25)10^{-6} = -300 \times 10^{-6}.$$

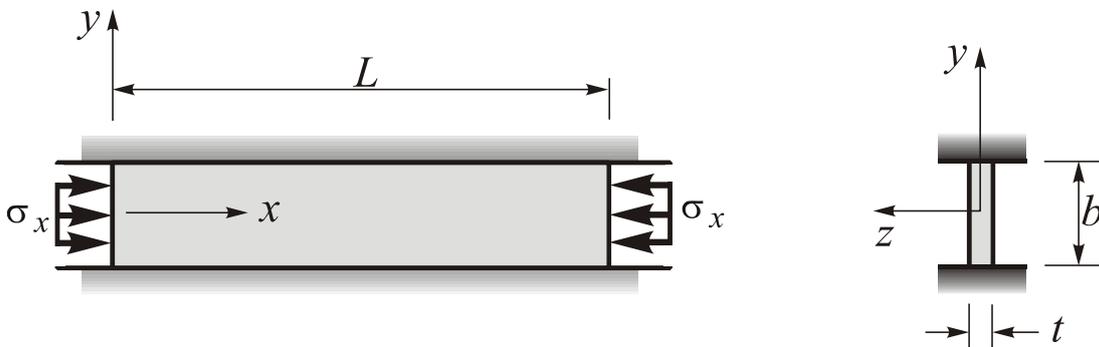
Therefore

$$\Delta V = eV_0 = (-300 \times 10^{-6})[\pi(60)^2(200)] = -679 \text{ mm}^3,$$

where the negative sign means a decrease in the volume of the cylinder.

### Example 4.3

A long, thin plate of thickness  $t$ , width  $b$ , and length  $L$  carries an axial load  $P$  which produces the uniform stress  $\sigma_x$ , as shown in figure. The edges at  $y = \pm b/2$  are placed between the two smooth, rigid walls so that lateral expansion in the  $y$  direction is prevented. Determine the components of stress and strain.



**Solution** In this example we have  $\gamma_{xy} = \gamma_{yz} = \gamma_{xz} = 0$ ,  $\varepsilon_y = 0$ ,  $\sigma_z = 0$ , and  $\sigma_x = -P/bt$ . Equations (4.10 through 4.15) then reduce to

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y), \quad (\text{a})$$

$$0 = \frac{1}{E}(\sigma_y - \nu\sigma_x), \quad (\text{b})$$

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \nu\sigma_y), \quad (\text{c})$$

from which

$$\sigma_y = \nu\sigma_x, \quad \varepsilon_x = \frac{1-\nu^2}{E}\sigma_x.$$

Substitution of the above into Eq. (c) results in  $\varepsilon_z = -\nu\varepsilon_x/(1-\nu)$ . We thus have

$$\sigma_x = -\frac{P}{bt}, \quad \sigma_y = -\nu\frac{P}{bt}, \quad \varepsilon_x = -\frac{1-\nu^2}{E}\frac{P}{bt}, \quad \varepsilon_z = \frac{\nu(1+\nu)}{E}\frac{P}{bt}.$$

It is interesting to note that the following ratios may now be formed:

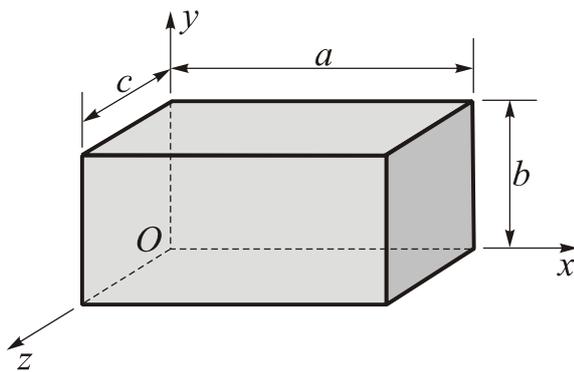
$$\frac{\sigma_x}{\varepsilon_x} = \frac{E}{1-\nu^2}, \quad -\frac{\varepsilon_z}{\varepsilon_x} = \frac{\nu}{1-\nu}.$$

The quantities  $E/(1-\nu^2)$  and  $\nu/(1-\nu)$  are called the *effective modulus of elasticity* and the *effective value of Poisson's ratio*, respectively. The former is useful in the theory of wide beams and plates.



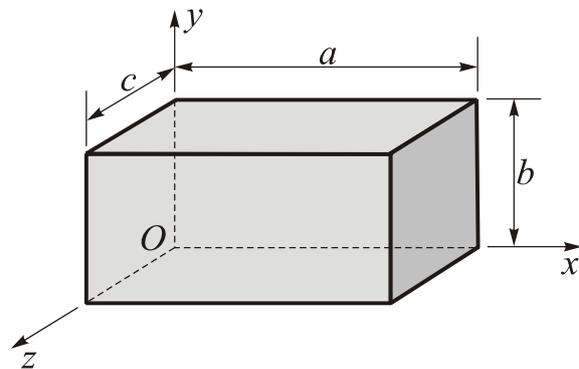
## PROBLEMS

**Problem 4.1** An element of aluminum in the form of a rectangular parallelepiped (see figure) of dimensions  $a = 3$  cm,  $b = 12$  cm, and  $c = 9$  cm is subjected to triaxial stresses  $\sigma_x = 75$  MPa,  $\sigma_y = -30$  MPa, and  $\sigma_z = -10$  MPa acting on the  $x$ ,  $y$ , and  $z$  faces, respectively. Determine the following quantities: (1) the maximum shear stress  $\tau_{\max}$  in the material; (2) the changes  $\Delta a$ ,  $\Delta b$ , and  $\Delta c$  in the dimensions of the element; (3) the change  $\Delta V$  in the volume; and (4) the strain energy  $U$  stored in the element. Assume  $E = 70$  GPa and  $\nu = 0.33$ .



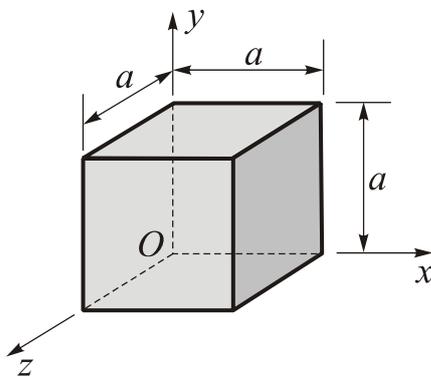
**Problem 4.2** An element of steel ( $E = 200$  GPa,  $\nu = 0.30$ ) in the form of a rectangular parallelepiped (see figure) of dimensions  $a = 250$  mm,

subjected to triaxial stresses  $\sigma_x = -50$  MPa,  $\sigma_y = -60$  MPa, and  $\sigma_z = -30$  MPa acting on the  $x$ ,  $y$ , and  $z$  faces, respectively. Determine the following quantities: (1) the maximum shear stress  $\tau_{\max}$  in the material; (2) the changes  $\Delta a$ ,  $\Delta b$ , and  $\Delta c$  in the dimensions of the element; (3) the change  $\Delta V$  in the volume; and (4) the strain energy  $U$  stored in the element.

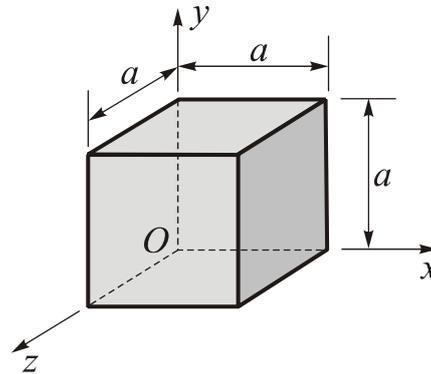


**Problem 4.3** A cube of cast iron ( $E = 95$  GPa and  $\nu = 0.25$ ) with sides of length  $a = 8$  cm (see figure) is tested in a laboratory under triaxial stress. Gages mounted on the testing machine show that the compressive strains in the material are  $\epsilon_x = -350 \times 10^{-6}$  and

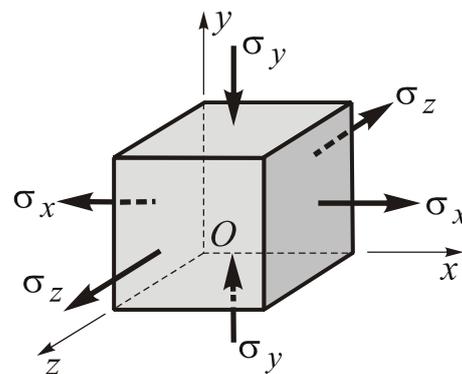
$\varepsilon_y = \varepsilon_z = -65 \times 10^{-6}$ . Determine the following quantities: (1) the normal stresses  $\sigma_x$ ,  $\sigma_y$  and  $\sigma_z$  acting on the  $x$ ,  $y$ , and  $z$  faces of the cube; (2) the maximum shear stress  $\tau_{\max}$  in the material; (3) the change  $\Delta V$  in the volume of the cube; and (4) the strain energy  $U$  stored in the cube.



**Problem 4.4** A cube of granite ( $E = 60$  GPa and  $\nu = 0.25$ ) with sides of length  $a = 60$  mm (see figure) is tested in a laboratory under triaxial stress. Gages mounted on the testing machine show that the compressive strains in the material are  $\varepsilon_x = -620 \times 10^{-6}$  and  $\varepsilon_y = \varepsilon_z = -250 \times 10^{-6}$ . Determine the following quantities: (1) the normal stresses  $\sigma_x$ ,  $\sigma_y$  and  $\sigma_z$  acting on the  $x$ ,  $y$ , and  $z$  faces of the cube; (2) the maximum shear stress  $\tau_{\max}$  in the material; (3) the change  $\Delta V$  in the volume of the cube; and (4) the strain energy  $U$  stored in the cube.

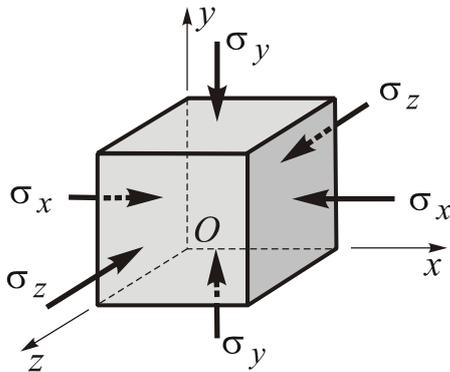


**Problem 4.5** An element of aluminum in triaxial stress (see figure) is subjected to stresses  $\sigma_x = 40$  MPa,  $\sigma_y = -35$  MPa, and  $\sigma_z = -20$  MPa. It is also known that the normal strains in the  $x$  and  $y$  directions are  $\varepsilon_x = 713.8 \times 10^{-6}$  (elongation) and  $\varepsilon_y = -502.3 \times 10^{-6}$  (shortening). What is the bulk modulus  $K$  for the aluminum?

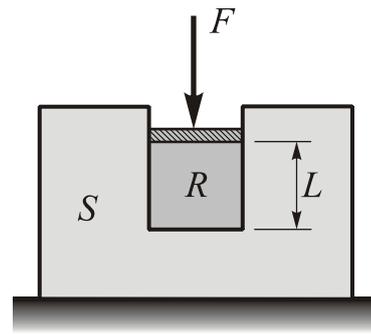
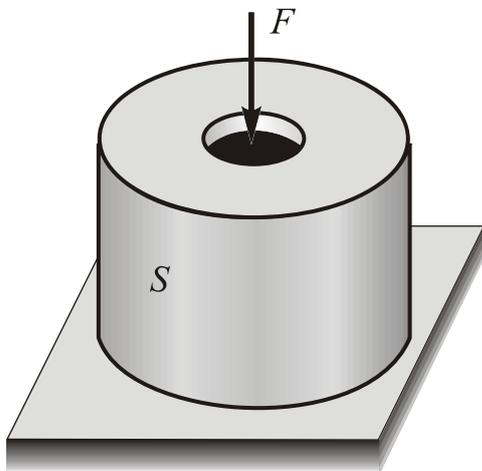


**Problem 4.6** An element of nylon in triaxial stress (see figure) is subjected to stresses  $\sigma_x = -4.5$  MPa,  $\sigma_y = -3.6$  MPa, and  $\sigma_z = -2.1$  MPa. It is also known that the normal strains in the  $x$  and  $y$  directions are

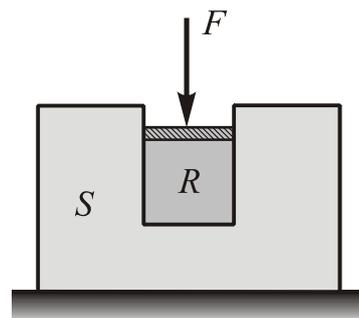
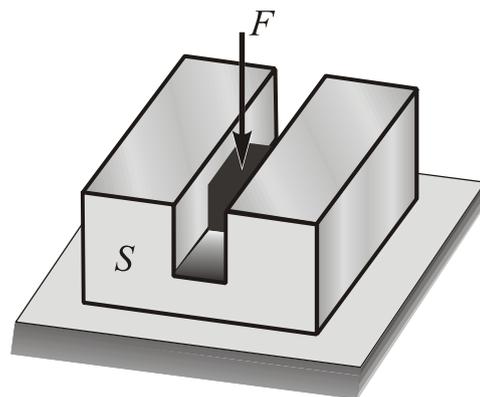
$\varepsilon_x = -740 \times 10^{-6}$  and  $\varepsilon_y = -320 \times 10^{-6}$  (shortenings). What is the bulk modulus  $K$  for the aluminum?



**Problem 4.7** A rubber cylinder  $R$  of length  $L$  and cross-sectional area  $A$  is compressed inside a steel cylinder  $S$  by a force  $F$  that applies a uniformly distributed pressure  $p_0$  to the rubber (see figure). (1) Derive a formula for the lateral pressure  $p$  between the rubber and the steel, disregarding friction between the rubber and the steel, and assuming that the steel cylinder is rigid when compared to the rubber; (2) Derive a formula for the shortening  $\delta$  of the rubber cylinder.



**Problem 4.8** A block  $R$  of rubber is confined between plane parallel walls of a steel block  $S$  (see figure). A uniformly distributed pressure  $p_0$  is applied to the top of the rubber block by a force  $F$ . (1) Derive a formula for the lateral pressure  $p$  between the rubber and the steel, disregarding friction between the rubber and the steel, and assuming that the steel block is rigid when compared to the rubber. (2) Derive a formula for the dilatation  $e$  of the rubber. (3) Derive a formula for the strain-energy density  $U_0$  of the rubber.



**Problem 4.9** A solid spherical ball of brass ( $E = 105 \text{ GPa}$ ,  $\nu = 0.34$ ) is lowered into the ocean to a depth of 5000 m. The diameter of the ball is 25 cm. Determine the decrease  $\Delta d$  in diameter, the decrease  $\Delta V$  in volume, and the strain energy  $U$  of the ball.

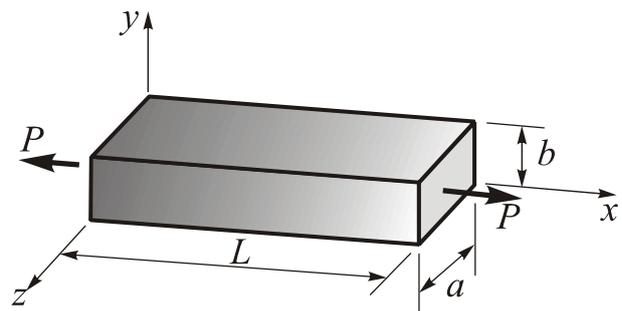
**Problem 4.10** A solid steel sphere ( $E = 200 \text{ GPa}$ ,  $\nu = 0.3$ ) is subjected to hydrostatic pressure  $p$  such that its volume is reduced by 0.4%. (1) Calculate the pressure  $p$ . (2) Calculate the volume modulus of elasticity  $K$  for the steel. (3) Calculate the strain energy  $U$  stored in the sphere if its diameter is  $d = 120 \text{ mm}$ .

**Problem 4.11** A solid bronze sphere (volume modulus of elasticity  $K = 100 \text{ GPa}$ ) is suddenly heated around its outer surface. The tendency of the heated part of the sphere to expand produces uniform tension in all directions at the center of the sphere. If the stress at the center is 80 MPa, what is the strain? Also, calculate the unit volume change  $e$  and the strain-energy density  $U_0$  at the center.

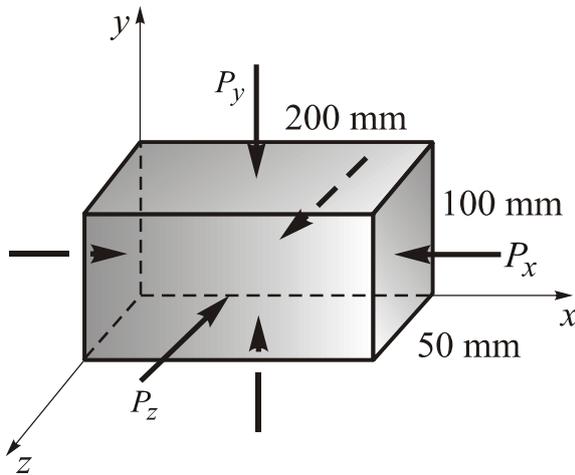
**Problem 4.12** A cube of magnesium 100 mm on each side is lowered into the ocean to a depth such that the length of each side shortens by

0.018 mm. Assuming that  $E = 45 \text{ GPa}$  and  $\nu = 0.35$ , calculate the following quantities: (1) the depth  $d$  to which the cube is lowered, and (2) the percent increase in the density of the magnesium.

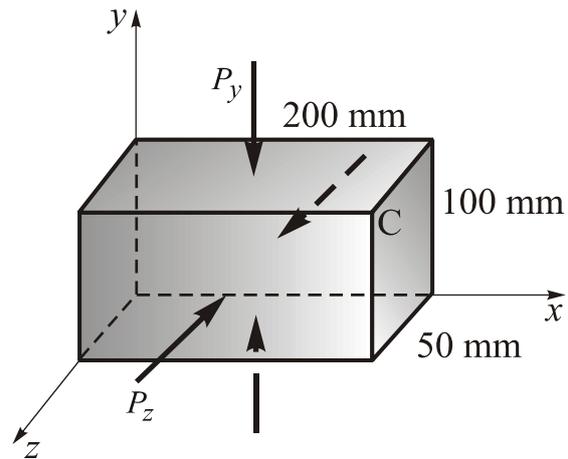
**Problem 4.13** Determine the axial strain in the block of the figure when subjected to an axial load 20 kN. The block is constrained against  $y$ - and  $z$ -directed contractions. Use  $a = 6.0 \text{ mm}$ ,  $b = 10.0 \text{ mm}$ ,  $L = 10.0 \text{ cm}$ ,  $E = 70 \text{ GPa}$ , and  $\nu = 0.33$ .



**Problem 4.14** The rectangular concrete block shown in the figure is subjected to loads which have the resultants  $P_x = 100 \text{ kN}$ ,  $P_y = 150 \text{ kN}$ , and  $P_z = 50 \text{ kN}$ . Calculate (1) the changes in lengths of the block and (2) the value of a single force system of compressive forces applied only on the  $y$  faces that would produce the same  $y$ -directed strain as do the initial forces. Use  $E = 24 \text{ GPa}$  and  $\nu = 0.2$ .



**Problem 4.15** The rectangular concrete block shown in the figure is subjected to loads which have the resultants  $P_x = 0$ ,  $P_y = 150$  kN, and  $P_z = 50$  kN. Calculate (1) the changes in lengths of the block and (2) the value of a single force system of compressive forces applied only on the  $y$  faces that would produce the same  $y$ -directed strain as do the initial forces. Use  $E = 24$  GPa and  $\nu = 0.2$ .



**Problem 4.16** A 50-mm-diameter and 100-mm-long solid cylinder is subjected to hydrostatic loading with  $\sigma_x = \sigma_y = \sigma_z = -50$  MPa. Use  $E = 205$  GPa and  $\nu = 0.33$ . Calculate (1) the change in length of the cylinder and (2) the change in volume of the cylinder.

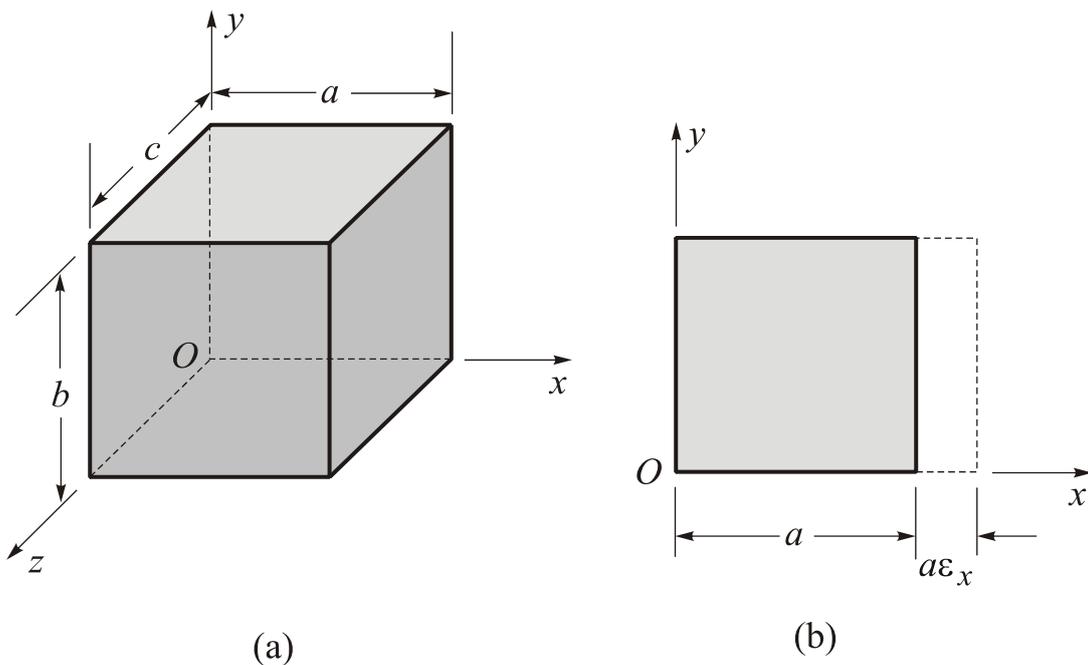
## Chapter 5 Plane Strain

The strains at a point in a deformable elastic solid vary according to the orientation of the axes, in a manner similar to that for stresses. First of all, we will derive the *transformation equations that relate the strains in inclined directions to the strains in the reference directions*. These *transformation equations are widely used in experimental investigations of stress-strain state involving measurements of strains*.

Strains are usually measured by *strain gages*; for example, gages are placed in aircraft to measure structural behavior during flight. Since each gage measures the strain in one particular direction, it is usually necessary to calculate the strains in other directions by means of the transformation equations.

### 5.1 Plane Strain versus Plane Stress Relations

First of all, let us estimate the term “*plane strain*” and its relations to plane stress. Consider a small element of material having sides of lengths  $a$ ,  $b$ , and  $c$  in the  $x$ ,  $y$ , and  $z$  directions, respectively (Fig. 5.1a). If the only deformations are those in the  $xy$  plane, then three strain components may exist: the normal strain  $\epsilon_x$  in the  $x$  direction (Fig. 5.1b), the normal strain  $\epsilon_y$  in the  $y$  direction (Fig. 5.1c), and the shear strain  $\gamma_{xy}$  (Fig. 5.1d). An element of material subjected to these strains is said to be in a *state of plane strain*. It follows that an element in plane strain has no normal strain  $\epsilon_z$  in the  $z$  direction and no shear strains  $\gamma_{xz}$  and  $\gamma_{yz}$  in the  $xz$  and  $yz$  planes, respectively.



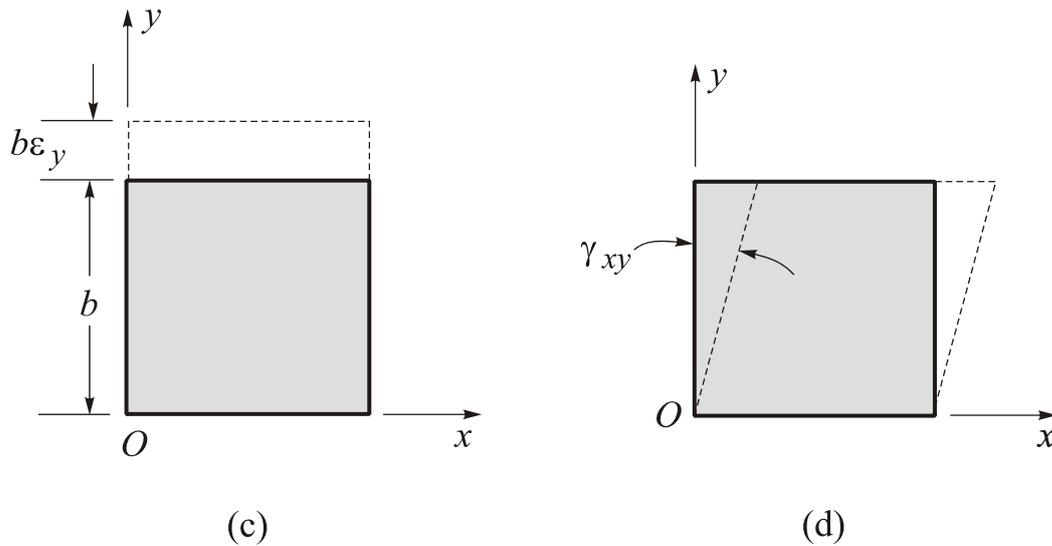


Fig. 5.1 Strain components  $\varepsilon_x$ ,  $\varepsilon_y$  and  $\gamma_{xy}$  in the  $xy$  plane (plane strain)

In result, plane strain is defined by the following conditions:

$$\varepsilon_z = 0, \quad \gamma_{xz} = 0, \quad \gamma_{yz} = 0. \quad (5.1)$$

The remaining strains ( $\varepsilon_x$ ,  $\varepsilon_y$  and  $\gamma_{xy}$ ) may have nonzero values.

The plane strain occurs when the *front and rear faces of an element of material* (Fig. 5.1a) are fully restrained against displacement. This idealized condition is seldom reached in real structural elements. But the transformation equations of plane strain are useful, because they also apply to plane stress.

The definition of plane strain (Eqs. 5.1) is analogous to that for plane stress. In plane stress, the following stresses must be zero:

$$\sigma_z = 0, \quad \tau_{xz} = 0, \quad \tau_{yz} = 0, \quad (5.2)$$

whereas the remaining stresses ( $\sigma_x$ ,  $\sigma_y$ , and  $\tau_{xy}$ ) may have nonzero values. A comparison of the stresses and strains in plane stress and plane strain is given in Fig. 5.2.

Under ordinary conditions plane stress and plane strain do not occur simultaneously. An element in plane stress will undergo a strain in the  $z$  direction (Fig. 5.2); hence, it is not in plane strain. Also, an element in plane strain usually will have stresses  $\sigma_z$  acting on it because of the requirement that  $\varepsilon_z = 0$ ; therefore, it is not in plane stress. An exception occurs when an element in plane stress is subjected to equal and opposite normal stresses (that is, when  $\sigma_x = -\sigma_y$ ) and Hooke's law holds for the material. In this special case, there is no normal strain in the  $z$  direction, as shown by Eq. (3.49), and therefore the element is in a state of plane strain as well as plane stress. Another hypothetical special case is

when a material has Poisson's ratio equal to zero ( $\nu = 0$ ). Then every plane stress element is also in plane strain because  $\epsilon_z = 0$  (Eq. 3.49).

The stress-transformation equations derived for plane stress in the  $xy$  plane (Eqs. 3.8 and 3.9) are valid even when a normal stress  $\sigma_z$  is present. The explanation is grounded on the fact that the stress  $\sigma_z$  does not enter the equations of equilibrium used in deriving Eqs. (3.8) and (3.9). Therefore, *the transformation equations for plane stress can also be used for the stresses in plane strain.*

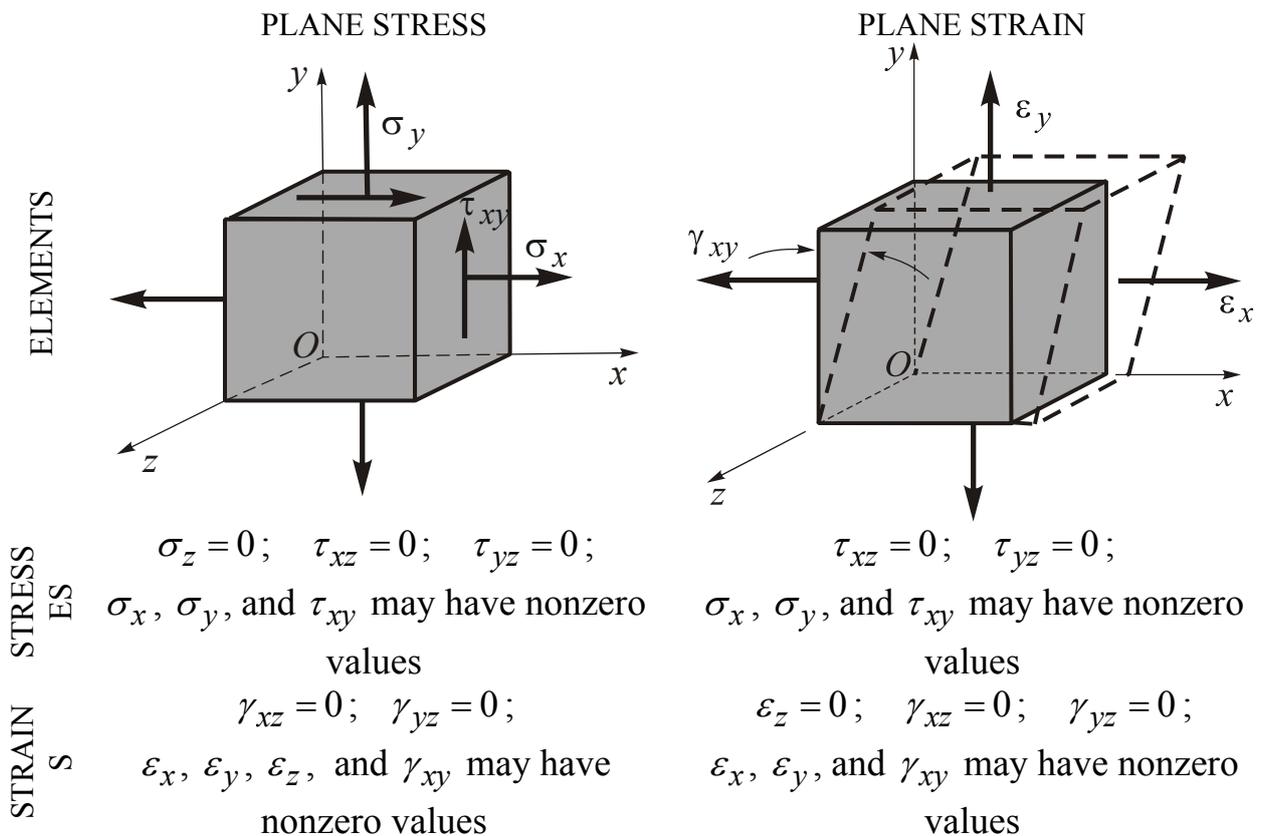


Fig. 5.2 Comparison of plane stress and plane strain

Now we will derive the strain-transformation equations for the case of plane strain in the  $xy$  plane. But these equations are valid even when a strain  $\epsilon_z$  exists. The reason is simple enough – the strain  $\epsilon_z$  does not affect the geometric relationships used in the derivations. Therefore, *the transformation equations for plane strain(ed) state can also be used for the strains in plane stress(ed) state.*

If you remember, the *transformation equations for plane stress state* were derived solely from equilibrium and therefore *are valid for any material*, whether linearly elastic or not. The same conclusion applies to the *transformation equations for plane strain state* since they are derived solely from geometry. *They are independent of the material properties.*

## 5.2 Transformation Equations for Plane Strain State

In the derivation of the transformation equations for plane strain, we will use the coordinate axes shown in Fig. 5.3. We will assume that the normal strains  $\varepsilon_x$  and  $\varepsilon_y$  and the shear strain  $\gamma_{xy}$  associated with the  $xy$  axes are known (Fig. 5.1). We will determine the normal strain  $\varepsilon_{x_1}$  and the shear strain  $\gamma_{x_1y_1}$  associated with the  $x_1y_1$  axes, which are rotated counterclockwise through positive angle  $\theta$  from the  $xy$  axes. It is not necessary to derive a separate equation for the normal strain  $\varepsilon_{y_1}$  because it can be obtained from the equation for  $\varepsilon_{x_1}$  by  $\theta + 90^\circ$  substituting for  $\theta$ .

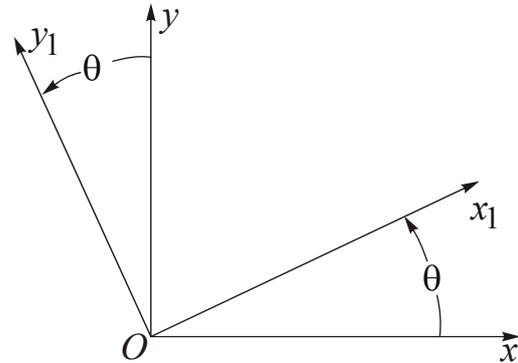


Fig. 5.3 Axes  $x_1$  and  $y_1$  rotated through an angle  $\theta$  from the  $xy$  axes

### 5.2.1 Normal strain $\varepsilon_{x_1}$

To determine the normal strain  $\varepsilon_{x_1}$  in the  $x_1$  direction, we consider infinitesimally small element of material oriented so that the  $x_1$  axis is along a diagonal of the  $z$  face of the element and the  $x$  and  $y$  axes are along the sides of the element (Fig. 5.4a). The figure shows a two-dimensional view of the three-dimensional element, with the  $z$  axis toward the viewer.

We will begin from the strain  $\varepsilon_x$  in the  $x$  direction (Fig. 5.4a). This strain produces an elongation in the  $x$  direction equal to  $\varepsilon_x dx$ , where  $dx$  is the length of the corresponding side of the element. As a result of this elongation, the diagonal of the element increases in length by an amount

$$\varepsilon_x dx \cos \theta, \quad (a)$$

as shown in Fig. 5.4a.

Secondly, consider the strain  $\varepsilon_y$  in the  $y$  direction (Fig. 5.4b). This strain produces an elongation in the  $y$  direction equal to  $\varepsilon_y dy$ , where  $dy$  is the length of the side of the element parallel to the  $y$  axis. As a result of this elongation, the diagonal of the element increases in length by an amount

$$\varepsilon_y dy \sin \theta, \quad (b)$$

which is shown in Fig. 5.4b.

Finally, consider the shear strain  $\gamma_{xy}$  in the  $xy$  plane (Fig. 5.4c). This strain produces a distortion of the element such that the angle at the lower left corner of the element decreases by an amount equal to the shear strain. Consequently, the upper face of the element moves to the right (with respect to the lower face) by an amount  $\gamma_{xy}dy$ . This deformation results in an increase in the length of the diagonal equal to

$$\gamma_{xy}dy \cos \theta, \tag{c}$$

as shown in Fig. 5.4c.

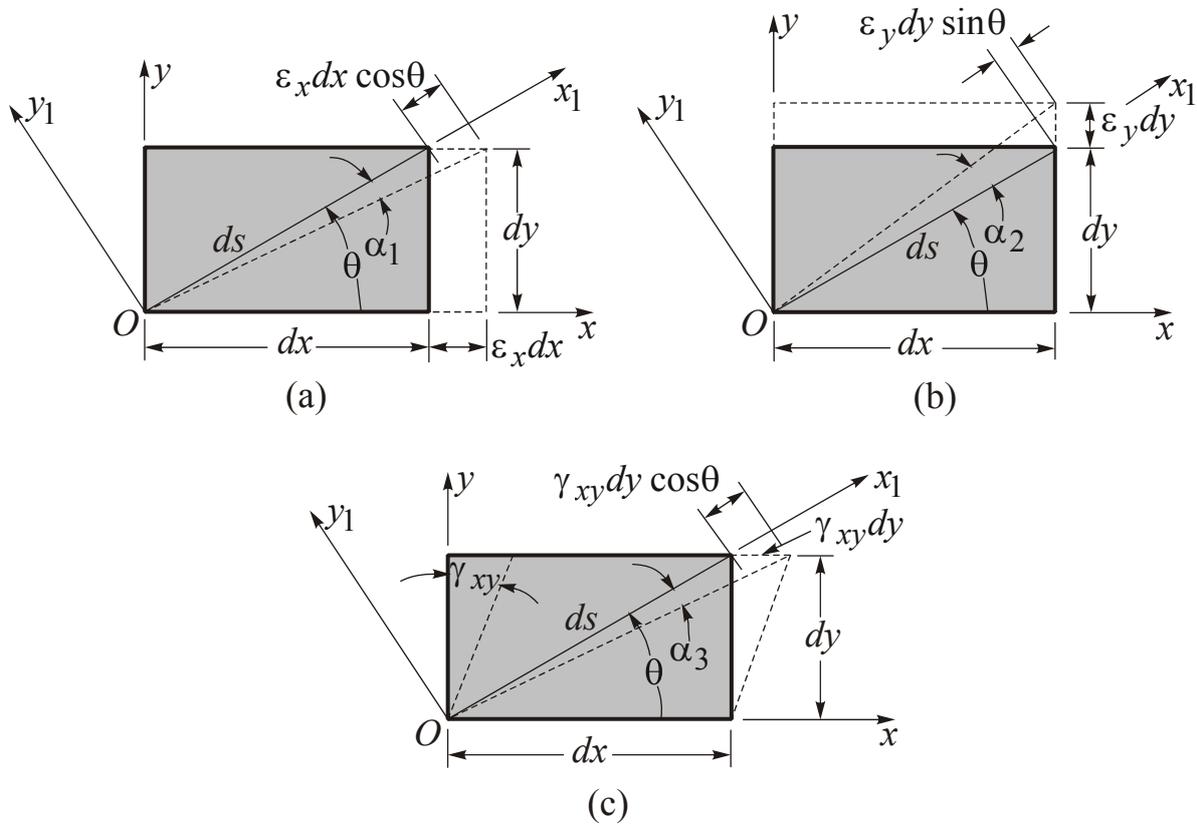


Fig. 5.4 Deformations of an element in plane strain due to: (a) normal strain  $\epsilon_x$ ; (b) normal strain  $\epsilon_y$ ; and (c) shear strain  $\gamma_{xy}$

The total increase  $\Delta ds$  in the length of the diagonal  $ds$  is the sum of the preceding three expressions; thus,

$$\Delta ds = \epsilon_x dx \cos \theta + \epsilon_y dy \sin \theta + \gamma_{xy} dy \cos \theta. \tag{d}$$

The normal strain  $\epsilon_{x_1}$  in the  $x_1$  direction is equal to this increase in length divided by the initial length  $ds$  of the diagonal:

$$\epsilon_{x_1} = \frac{\Delta ds}{ds} = \epsilon_x \frac{dx}{ds} \cos \theta + \epsilon_y \frac{dy}{ds} \sin \theta + \gamma_{xy} \frac{dy}{ds} \cos \theta. \tag{e}$$

Taking into account that  $dx/ds = \cos\theta$  and  $dy/ds = \sin\theta$ , we obtain the following equation for the normal strain:

$$\varepsilon_{x_1} = \varepsilon_x \cos^2 \theta + \varepsilon_y \sin^2 \theta + \gamma_{xy} \sin \theta \cos \theta. \quad (5.3)$$

Thus, we have obtained an expression for the normal strain in the  $x_1$  direction in terms of the strains  $\varepsilon_x$ ,  $\varepsilon_y$ , and  $\gamma_{xy}$  associated with the  $xy$  axes. As mentioned previously, the normal strain  $\varepsilon_{y_1}$  in the  $y_1$  direction is obtained from this equation by substituting  $\theta + 90^\circ$  for  $\theta$ .

### 5.2.2 Shear strain $\gamma_{x_1y_1}$ .

Let us now find the shear strain  $\gamma_{x_1y_1}$  associated with the  $x_1y_1$  axes. This strain is equal to the decrease in angle between lines in the material that were initially along the  $x_1$  and  $y_1$  axes. Consider Fig. 5.5, which shows both the  $xy$  and  $x_1y_1$  axes, with the angle between them. The line  $Oa$  represents a line in the material that initially was along the  $x_1$  axis (that is, along the diagonal of the element in Fig. 5.4). The deformations produced by the strains  $\varepsilon_x$ ,  $\varepsilon_y$ , and  $\gamma_{xy}$  (Fig. 5.4) cause line  $Oa$  rotate through a counterclockwise angle  $\alpha$  from the  $x_1$  axis to the position shown in Fig. 5.5. Similarly, line  $Ob$  was originally along the  $y_1$  axis, but because of the deformations it rotates through a clockwise angle  $\beta$ . The shear strain  $\gamma_{x_1y_1}$  is the decrease in angle between the two lines that originally were at right angles:

$$\gamma_{x_1y_1} = \alpha + \beta. \quad (5.4)$$

In order to find the shear strain  $\gamma_{x_1y_1}$ , we must determine the angles  $\alpha$  and  $\beta$ .

The angle  $\alpha$  can be found from the deformations shown in Fig. 5.4 as follows. The strain  $\varepsilon_x$  (Fig. 5.4a) produces a clockwise rotation of the diagonal of the element. Let us denote this angle of rotation as  $\alpha_1$ . It is equal to the distance  $\varepsilon_x dx \sin\theta$  divided by the length  $ds$  of the diagonal:

$$\alpha_1 = \varepsilon_x \frac{dx}{ds} \sin \theta. \quad (f)$$

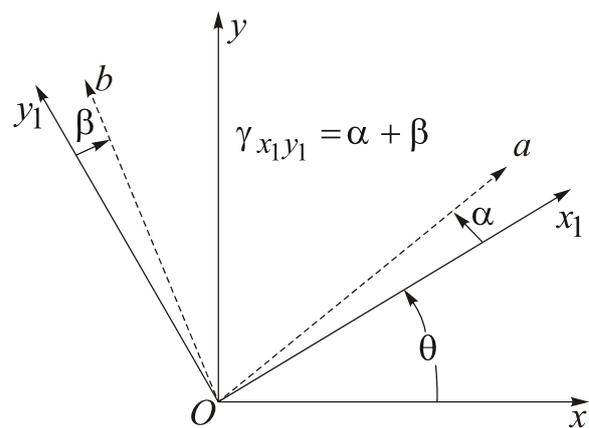


Fig. 5.5 Shear strain  $\gamma_{x_1y_1}$  associated with the  $x_1y_1$  axes

Similarly, the strain  $\varepsilon_y$  produces a counterclockwise rotation of the diagonal through an angle  $\alpha_2$  (Fig. 5.4b). This angle is equal to the distance  $\varepsilon_y dy \cos \theta$  divided by  $ds$  :

$$\alpha_2 = \varepsilon_y \frac{dy}{ds} \cos \theta. \quad (\text{g})$$

Finally, the strain  $\gamma_{xy}$  produces a clockwise rotation through an angle  $\alpha_3$  (Fig. 5.4c) equal to the distance  $\gamma_{xy} dy \sin \theta$  divided by  $ds$  :

$$\alpha_3 = \gamma_{xy} \frac{dy}{ds} \sin \theta. \quad (\text{h})$$

Therefore, the resultant counterclockwise rotation of the diagonal (Fig. 5.4), equal to the angle  $\alpha$  shown in Fig. 5.5, is

$$\alpha = -\alpha_1 + \alpha_2 - \alpha_3 = -\varepsilon_x \frac{dx}{ds} \sin \theta + \varepsilon_y \frac{dy}{ds} \cos \theta - \gamma_{xy} \frac{dy}{ds} \sin \theta. \quad (\text{i})$$

Again observing that  $dx/ds = \cos \theta$  and  $dy/ds = \sin \theta$ , we obtain

$$\alpha = -(\varepsilon_x - \varepsilon_y) \sin \theta \cos \theta - \gamma_{xy} \sin^2 \theta. \quad (5.5)$$

The rotation of line  $Ob$  (Fig. 5.5), which initially was at  $90^\circ$  to line  $Oa$ , can be found by substituting  $\theta + 90^\circ$  for  $\theta$  in the expression for  $\alpha$ . The resulting expression is counterclockwise when positive (because  $\alpha$  is counterclockwise when positive), hence it is equal to the negative of the angle  $\beta$  (because  $\beta$  is positive when clockwise). Thus,

$$\begin{aligned} \beta &= (\varepsilon_x - \varepsilon_y) \sin(\theta + 90^\circ) \cos(\theta + 90^\circ) + \gamma_{xy} \sin^2(\theta + 90^\circ) = \\ &= -(\varepsilon_x - \varepsilon_y) \sin \theta \cos \theta + \gamma_{xy} \cos^2 \theta. \end{aligned} \quad (5.6)$$

Adding  $\alpha$  and  $\beta$  gives the shear strain  $\gamma_{x_1 y_1}$  (see Eq. 5.4):

$$\gamma_{x_1 y_1} = -2(\varepsilon_x - \varepsilon_y) \sin \theta \cos \theta + \gamma_{xy} (\cos^2 \theta - \sin^2 \theta). \quad (\text{j})$$

To obtain more useful form, let us divide each term by 2:

$$\frac{\gamma_{x_1 y_1}}{2} = -(\varepsilon_x - \varepsilon_y) \sin \theta \cos \theta + \frac{\gamma_{xy}}{2} (\cos^2 \theta - \sin^2 \theta). \quad (5.7)$$

In result, we obtained the expression for the shear strain  $\gamma_{x_1 y_1}$  associated with the  $x_1 y_1$  axes in terms of the strains  $\varepsilon_x$ ,  $\varepsilon_y$ , and  $\gamma_{xy}$  associated with the  $xy$  axes.

### 5.2.3 Transformation Equations For Plane Strain

The equations for plane strain (Eqs. 5.3 and 5.7) can be expressed in terms of the angle  $2\theta$  by using the following trigonometric identities:

$$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta), \quad \sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta), \quad \sin \theta \cos \theta = \frac{1}{2} \sin 2\theta.$$

Thus, the *transformation equations for plane strain* are

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta, \quad (5.8)$$

$$\frac{\gamma_{x_1 y_1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2} \sin 2\theta + \frac{\gamma_{xy}}{2} \cos 2\theta. \quad (5.9)$$

**Note.** These equations are the counterparts of Eqs. (3.8) and (3.9) for plane stress.

In comparison of two sets of equations,  $\varepsilon_{x_1}$  corresponds to  $\sigma_{x_1}$ ,  $\gamma_{x_1 y_1}/2$  corresponds to  $\tau_{x_1 y_1}$ ,  $\varepsilon_x$  corresponds to  $\sigma_x$ ,  $\varepsilon_y$  corresponds to  $\sigma_y$  and  $\gamma_{xy}/2$  corresponds to  $\tau_{xy}$ .

The analogy between the transformation equations for plane stress and those for plane strain shows that all of the observations made in Chapter 3 concerning plane stress, principal stresses, maximum shear stresses, and Mohr's circle have their counterparts in plane strain. For instance, the sum of the normal strains in perpendicular directions is a constant (compare with Eq. 3.11):

$$\varepsilon_{x_1} + \varepsilon_{y_1} = \varepsilon_x + \varepsilon_y. \quad (5.10)$$

**Note.** This equality can be verified easily by substituting the expressions for  $\varepsilon_{x_1}$  (from Eq. 5.8) and  $\varepsilon_{y_1}$  (from Eq. 5.8 with  $\theta$  replaced by  $\theta + 90^\circ$ ).

### 5.3 Principal Strains

Principal strains exist on perpendicular planes with the principal angles  $\theta_p$  calculated from the following equation (compare with Eq. 3.19):

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}. \quad (5.11)$$

The principal strains can be calculated from the equation

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}, \quad (5.12)$$

which corresponds to Eq. (3.26) for the principal stresses. The two principal strains (in the  $xy$  plane) can be correlated with the two principal directions using the technique described for the principal stresses.

**Note. (1) In plane strain the third principal strain is  $\varepsilon_z = 0$ . (2) The shear strains are zero on the principal planes.**

### 5.4 Maximum Shear Strains

The maximum shear strains in the  $xy$  plane are associated with axes at  $45^\circ$  to the directions of the principal strains. The algebraically maximum shear strain (in the  $xy$  plane) is given by the following equation (compare with Eq. 3.38):

$$\frac{\gamma_{\max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}. \quad (5.13)$$

The algebraically minimum shear strain has the same magnitude but is negative.

In the directions of maximum shear strain, the normal strains are

$$\varepsilon_{aver} = \frac{\varepsilon_x + \varepsilon_y}{2} = \frac{\varepsilon_1 + \varepsilon_2}{2}. \quad (5.14)$$

Eq. (5.14) is analogous to Eq. (3.40) for stresses.

The *true maximum shearing strain of three-dimensional analysis* proceeds from Eq. (3.39):

$$(\gamma_{\max})_t = \varepsilon_1 - \varepsilon_3. \quad (5.15)$$

Here  $\varepsilon_1$  and  $\varepsilon_3$  are the algebraically largest and smallest principal strains, respectively.

The maximum out-of-plane shear strains, that is, the shear strains in the  $xz$  and  $yz$  planes, can be obtained from equations analogous to Eq. (5.13).

An element in plane stress that is oriented to the principal directions of stress (see Fig. 3.17b) has no shear stresses acting on its faces. Therefore, the shear strain  $\gamma_{x_1y_1}$  for this element is zero. It follows that the normal strains in this element are the principal strains. Thus, *at a given point in a stressed body, the principal strains and principal stresses occur in the same directions.*

### 5.5 Mohr's Circle for Plane Strain

Mohr's circle for plane strain is constructed in the same manner as the circle for plane stress, as illustrated in Fig. 5.6. Normal strain  $\varepsilon_{x_1}$  is plotted as the abscissa (positive to the right) and one-half the shear strain  $\left(\frac{\gamma_{x_1y_1}}{2}\right)$  is plotted as

the ordinate (positive downward). The center  $C$  of the circle has an abscissa equal to  $\varepsilon_{aver}$  (Eq. 5.14).

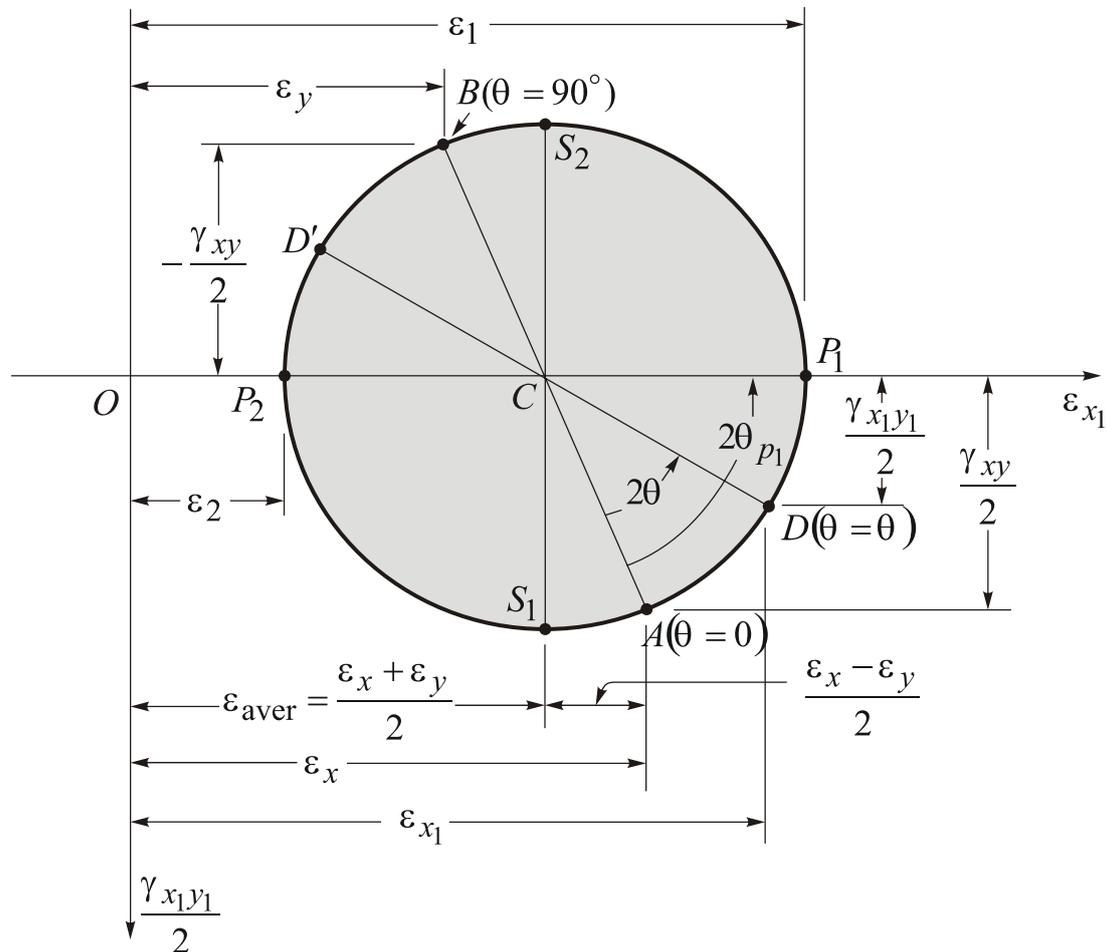


Fig. 5.6 Mohr's circle for plane strain

Point  $A$ , representing the strains associated with the  $x$  direction ( $\theta = 0$ ), has coordinates  $\varepsilon_x$  and  $\gamma_{xy}/2$ . Point  $B$ , at the opposite end of a diameter from  $A$ , has coordinates  $\varepsilon_y$  and  $-\gamma_{xy}/2$ , representing the strains associated with a pair of axes rotated through an angle  $\theta = 90^\circ$ .

The strains associated with axes rotated through an angle  $\theta$  are given by point  $D$ , which is located on the circle by measuring an angle  $2\theta$  counterclockwise from radius  $CA$ . The principal strains are represented by points  $P_1$  and  $P_2$  and the maximum shear strains by points  $S_1$  and  $S_2$ . All of these strains can be determined from the geometry of the circle or from the transformation equations.

### 5.6 Measurement of Strains (Continued)

Strain gages of different types and design as the most accurate method have been developed for measuring the normal strain state components *on the free surface* of any structural element *where the plane stress exists*. Now we consider a typical bonded strain gage and its special combinations.

Taking the outward normal to the surface as the  $z$  direction, we have  $\sigma_z = \tau_{yz} = \tau_{xz} = 0$ . Inasmuch as this stress condition offers no restraint to out-of-plane elastic deformation, a normal strain develops in addition to the in-plane strain components  $(\varepsilon_x, \varepsilon_y, \gamma_{xy})$ . It follows from the generalized Hook's law that  $\gamma_{xz} = \gamma_{yz} = 0$ . Thus the strain  $\varepsilon_z$  is a principal strain. This out-of-plane normal strain is significant in the determination of the *true maximum shear strain*. Observe that Eqs. (5.8, 5.9) were derived for a state of plane strain. However, the principal strain  $\varepsilon_z$  is obtained in terms of  $\varepsilon_x$  and  $\varepsilon_y$ , using equation, derived from Eqs. (3.55):

$$\varepsilon_z = -\frac{\nu}{1-\nu}(\varepsilon_x + \varepsilon_y). \quad (5.16)$$

Since each gage measures the normal strain in only one direction, and since the directions of the principal stresses are usually unknown, it is necessary to use three gages in combination, with each gage measuring the strain in a different direction. From three such measurements, it is possible to calculate the strains in any direction. *A group of three gages arranged in a particular pattern is called a strain rosette*. Because the rosette is mounted on the surface of the body, where the material is in plane stress, we can use the transformation equations for plane strain to calculate the strains in various directions.

A rosette usually consists of three specifically arranged gages whose axes are either  $45$  or  $60^\circ$  apart, as illustrated in the Fig. 5.7.

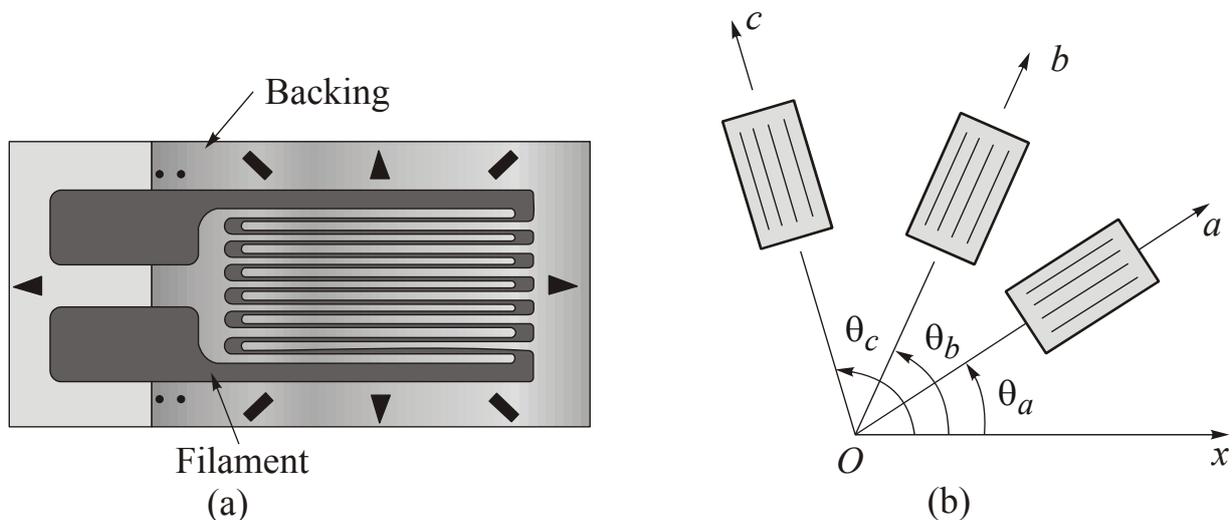


Fig. 5.7 (a) strain gage; (b) strain rosette

Consider three strain gages with angles  $\theta_a$ ,  $\theta_b$ , and  $\theta_c$  with respect to the reference  $x$  axis (Fig. 5.7b). The  $a$ -,  $b$ -, and  $c$ -directed normal strains are from Eq. (5.3):

$$\begin{aligned}\varepsilon_a &= \varepsilon_x \cos^2 \theta_a + \varepsilon_y \sin^2 \theta_a + \gamma_{xy} \sin \theta_a \cos \theta_a, \\ \varepsilon_b &= \varepsilon_x \cos^2 \theta_b + \varepsilon_y \sin^2 \theta_b + \gamma_{xy} \sin \theta_b \cos \theta_b, \\ \varepsilon_c &= \varepsilon_x \cos^2 \theta_c + \varepsilon_y \sin^2 \theta_c + \gamma_{xy} \sin \theta_c \cos \theta_c.\end{aligned}\quad (5.17)$$

When the values of  $\varepsilon_a$ ,  $\varepsilon_b$ ,  $\varepsilon_c$  are measured for prescribed  $\theta_a$ ,  $\theta_b$ , and  $\theta_c$ , the values of  $\varepsilon_x$ ,  $\varepsilon_y$ , and  $\gamma_{xy}$  can be found by simultaneous solution of Eqs. (5.17).

## 5.7 Calculation of Stresses

The strain equations are derived solely from geometry. Therefore, the equations apply to any material, whether linear or nonlinear, elastic or inelastic. However, *if it is desired to determine the stresses from the strains, the material properties must be taken into account.*

If the material follows Hooke's law, we can find the stresses using the appropriate stress-strain equations (for example, from Section 3.5 for plane stress or Section 4.2 for triaxial stress).

Suppose that the material is in plane stress and that we know the strains  $\varepsilon_x$ ,  $\varepsilon_y$ , and  $\gamma_{xy}$  from strain-gage measurements. Then we can use the stress-strain equations for plane stress (Eqs. 3.51 through 3.53) to obtain the stresses in the material.

Secondly suppose, that we have determined the three principal strains  $\varepsilon_1$ ,  $\varepsilon_2$ , and  $\varepsilon_3$  for an element of material (if the element is in plane strain, then  $\varepsilon_3 = 0$ ). Knowing these strains, we can find the principal stresses using Hooke's law for triaxial stress (see Eqs. 4.19 through 4.21). Once the principal stresses are known, we can find the stresses on inclined planes using the transformation equations for plane stress.



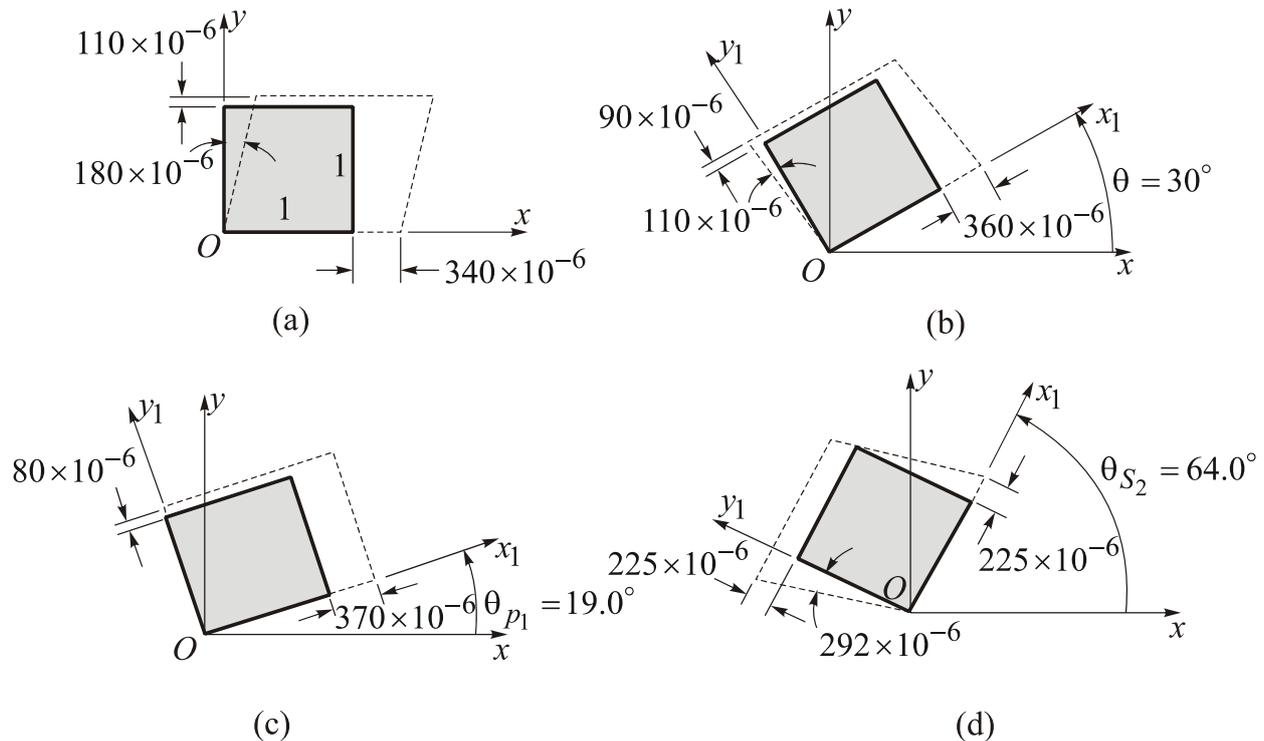
### EXAMPLES

#### Example 5.1

An element of material in plane strain undergoes the following strains:

$$\varepsilon_x = 340 \times 10^{-6}, \quad \varepsilon_y = 110 \times 10^{-6}, \quad \gamma_{xy} = 180 \times 10^{-6}.$$

These strains are shown in figure (a), as the deformations of an element of unit dimensions. Since the edges of the element have unit lengths, the changes in linear dimensions have the same magnitudes as the normal strains  $\varepsilon_x$  and  $\varepsilon_y$ . The shear strain  $\gamma_{xy}$  is the decrease in angle at the lower-left corner of the element.



Element of material in plane strain: (a) element oriented to the  $x$  and  $y$  axes; (b) element oriented at an angle  $\theta = 30^\circ$ ; (c) principal strains; (d) maximum shear strains. The edges of the elements have unit lengths

Determine the following quantities: (1) the strains for an element oriented at an angle  $\theta = 30^\circ$ , (2) the principal strains, and (3) the maximum shear strains. (Consider only the in-plane strains, and show all results on sketches of properly oriented elements).

**Solution** (1) *Element oriented at an angle  $\theta = 30^\circ$ .* The strains for an element oriented at an angle  $\theta$  to the  $x$  axis can be found from the transformation equations (Eqs. 5.8 and 5.9) after preliminary calculations:

$$\frac{\varepsilon_x + \varepsilon_y}{2} = \frac{(340 + 110) \times 10^{-6}}{2} = 225 \times 10^{-6},$$

$$\frac{\varepsilon_x - \varepsilon_y}{2} = \frac{(340 - 110) \times 10^{-6}}{2} = 115 \times 10^{-6},$$

$$\frac{\gamma_{xy}}{2} = 90 \times 10^{-6}.$$

Now substituting into Eqs. (5.8) and (5.9), we get

$$\begin{aligned}\varepsilon_{x_1} &= \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta = \\ &= (225 \times 10^{-6}) + (115 \times 10^{-6}) (\cos 60^\circ) + (90 \times 10^{-6}) (\sin 60^\circ) = 360 \times 10^{-6}, \\ \frac{\gamma_{x_1y_1}}{2} &= -\frac{\varepsilon_x - \varepsilon_y}{2} \sin 2\theta + \frac{\gamma_{xy}}{2} \cos 2\theta = \\ &= -(115 \times 10^{-6}) (\sin 60^\circ) + (90 \times 10^{-6}) (\cos 60^\circ) = -55 \times 10^{-6}.\end{aligned}$$

Therefore, the shear strain is

$$\gamma_{x_1y_1} = -110 \times 10^{-6}.$$

The strain  $\varepsilon_{y_1}$  can be obtained from Eq. (5.10), as follows:

$$\varepsilon_{y_1} = \varepsilon_x + \varepsilon_y - \varepsilon_{x_1} = (340 + 110 - 360) \times 10^{-6} = 90 \times 10^{-6}.$$

The strains  $\varepsilon_{x_1}$ ,  $\varepsilon_{y_1}$  and  $\gamma_{x_1y_1}$  are shown in figure (b) for an element oriented at  $\theta = 30^\circ$ . **Note that the angle at the lower-left corner of the element increases because  $\gamma_{x_1y_1}$  is negative.**

(2) *Principal strains.* The principal strains are readily determined from Eq. (5.12), as follows:

$$\begin{aligned}\varepsilon_{1,2} &= \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = \\ &= 225 \times 10^{-6} \pm \sqrt{(115 \times 10^{-6})^2 + (90 \times 10^{-6})^2} = 225 \times 10^{-6} \pm 146 \times 10^{-6}.\end{aligned}$$

In result, the principal strains are

$$\varepsilon_1 = 370 \times 10^{-6}, \quad \varepsilon_2 = 80 \times 10^{-6},$$

in which  $\varepsilon_1$  denotes the algebraically larger principal strain and  $\varepsilon_2$  denotes the algebraically smaller principal strain (keep in mind, we consider only in-plane strains).

The angles to the principal directions can be obtained from Eq. (5.11):

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = \frac{180}{340 - 110} = 0.7826.$$

The values of  $2\theta_p$  between 0 and  $360^\circ$  are  $38.0^\circ$  and  $218.0^\circ$ , and therefore the angles to the principal directions are

$$\theta_p = 19.0^\circ \text{ and } 109.0^\circ.$$

To determine the value of  $\theta_p$  associated with each principal strain, we substitute  $\theta_p = 19.0^\circ$  into the first transformation equation (Eq. 5.8) and solve for the strain:

$$\begin{aligned}\varepsilon_{x_1} &= \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta = \\ &= (225 \times 10^{-6}) + (115 \times 10^{-6}) (\cos 38^\circ) + (90 \times 10^{-6}) (\sin 38.0^\circ) = 370 \times 10^{-6}.\end{aligned}$$

This result shows that the larger principal strain  $\varepsilon_1$  is at the angle  $\theta_{p_1} = 19.0^\circ$ .

The smaller strain  $\varepsilon_2$  acts at  $90^\circ$  from that direction ( $\theta_{p_2} = 109.0^\circ$ ). Thus,

$$\begin{aligned}\varepsilon_1 &= 370 \times 10^{-6} \text{ and } \theta_{p_1} = 19.0^\circ \quad (\varepsilon_1 > \varepsilon_{x_1}, \text{ i.e. } 370 \times 10^{-6} > 360 \times 10^{-6}), \\ \varepsilon_2 &= 80 \times 10^{-6} \text{ and } \theta_{p_2} = 109.0^\circ \quad (\varepsilon_2 < \varepsilon_{y_1}, \text{ i.e. } 80 \times 10^{-6} < 90 \times 10^{-6}).\end{aligned}$$

**Note that**  $\varepsilon_1 + \varepsilon_2 = \varepsilon_x + \varepsilon_y$ :  $(370 \times 10^{-6} + 80 \times 10^{-6}) = (360 \times 10^{-6} + 90 \times 10^{-6})$ .

The principal strains are shown in figure (c). **Note, that there are no shear strains on the principal planes.**

(3) *Maximum shear strain.* The maximum shear strain is calculated from Eq. (5.13):

$$\begin{aligned}\frac{\gamma_{\max}}{2} &= \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = \sqrt{(115 \times 10^{-6})^2 + (90 \times 10^{-6})^2} = 146 \times 10^{-6}, \\ \gamma_{\max} &= 292 \times 10^{-6}.\end{aligned}$$

The element having the maximum shear strains is oriented at  $45^\circ$  to the principal directions; therefore,  $\theta_s = 19.0^\circ + 45^\circ = 64.0^\circ$  and  $2\theta_s = 128.0^\circ$ . By substituting this value of  $2\theta_s$  into the second transformation equation (Eq. 5.9), we can determine the sign of the shear strain associated with this direction. The calculations are as follows:

$$\begin{aligned}\frac{\gamma_{x_1y_1}}{2} &= -\frac{\varepsilon_x - \varepsilon_y}{2} \sin 2\theta + \frac{\gamma_{xy}}{2} \cos 2\theta = \\ &= -(115 \times 10^{-6}) (\sin 128.0^\circ) + (90 \times 10^{-6}) (\cos 128.0^\circ) = -146 \times 10^{-6}.\end{aligned}$$

This result shows that an element oriented at an angle  $\theta_{s_2} = 64.0^\circ$  has the maximum negative shear strain.

We can get the same result by observing that the angle  $\theta_{s_1}$  to the direction of maximum positive shear strain is always  $45^\circ$  less than  $\theta_{p_1}$ . Hence,

$$\theta_{s_1} = \theta_{p_1} - 45^\circ = 19.0^\circ - 45^\circ = -26.0^\circ,$$

$$\theta_{s_2} = \theta_{s_1} + 90^\circ = 64.0^\circ.$$

The shear strains corresponding to  $\theta_{s_1}$  and  $\theta_{s_2}$  are  $\gamma_{\max} = 292 \times 10^{-6}$  and  $\gamma_{\min} = -292 \times 10^{-6}$ , respectively.

The normal strains on the element having the maximum and minimum shear strains are

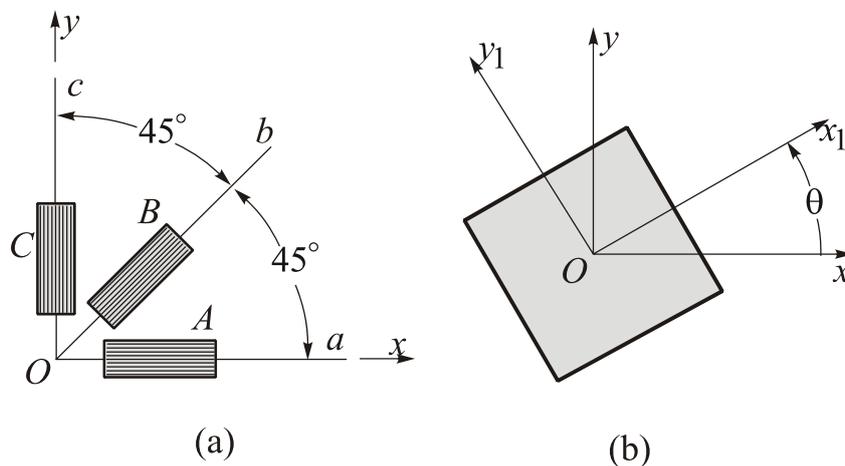
$$\varepsilon_{\text{aver}} = \frac{\varepsilon_x + \varepsilon_y}{2} = 225 \times 10^{-6}.$$

A sketch of the element having the maximum in-plane shear strains is shown in figure (d).

### Example 5.2

A  $45^\circ$  strain rosette consists of three electrical-resistance strain gages arranged to measure strains in two perpendicular directions and also at a  $45^\circ$  angle between them, as shown in figure (a). The rosette is bonded to the surface of the structure. Gages  $A$ ,  $B$ , and  $C$  measure the normal strains  $\varepsilon_a$ ,  $\varepsilon_b$ , and  $\varepsilon_c$  in the directions of lines  $Oa$ ,  $Ob$ , and  $Oc$ , respectively.

Explain how to obtain the strains  $\varepsilon_{x_1}$ ,  $\varepsilon_{y_1}$ , and  $\gamma_{x_1y_1}$  associated with an element oriented at an angle  $\theta$  to the  $xy$  axes (figure (b)).



**Solution** At the surface of the stressed object, the material is in plane stress. Since the strain transformation equations (Eqs. 5.8 and 5.9) apply to plane stress as well as to plane strain, we can use those equations to determine the strains in any direction.

(1) *Strains associated with the  $xy$  axes.* Because gages  $A$  and  $C$  are aligned with the  $x$  and  $y$  axes, respectively, they give the strains  $\varepsilon_x$  and  $\varepsilon_y$  directly:

$$\varepsilon_x = \varepsilon_a, \quad \varepsilon_y = \varepsilon_c.$$

To obtain the shear strain  $\gamma_{xy}$ , we use the transformation equation for normal strains (Eq. 5.8):

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta.$$

For an angle  $\theta = 45^\circ$ , we know that  $\varepsilon_{x_1} = \varepsilon_b$  (figure (a)); therefore, the preceding equation gives

$$\varepsilon_b = \frac{\varepsilon_a + \varepsilon_c}{2} + \frac{\varepsilon_a - \varepsilon_c}{2} (\cos 90^\circ) + \frac{\gamma_{xy}}{2} (\sin 90^\circ).$$

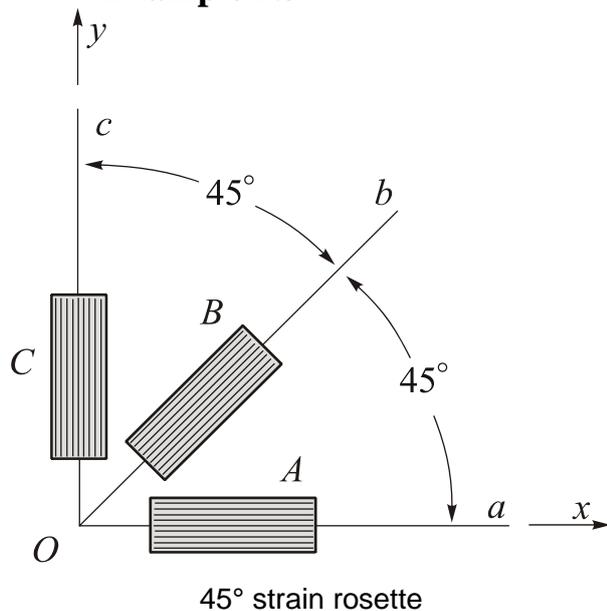
Solving for  $\gamma_{xy}$ , we get

$$\gamma_{xy} = 2\varepsilon_b - \varepsilon_a - \varepsilon_c.$$

Thus, the strains  $\varepsilon_x$ ,  $\varepsilon_y$  and  $\gamma_{xy}$  may be determined from the given strain-gage readings.

(2) *Strains associated with the  $x_1y_1$  axes.* Knowing the strains  $\varepsilon_x$ ,  $\varepsilon_y$  and  $\gamma_{xy}$ , we can calculate the strains for an element oriented at any angle  $\theta$  (figure (b)) from the strain-transformation equations (Eqs. 5.8 and 5.9) or from Mohr's circle. We can also calculate the principal strains and the maximum shear strains from Eqs. (5.12) and (5.13), respectively.

### Example 5.3



Using a  $45^\circ$  rosette, the following strains are measured at a point on the free surface of a stressed member:

$$\varepsilon_a = 900 \times 10^{-6},$$

$$\varepsilon_b = 700 \times 10^{-6},$$

$$\varepsilon_c = -100 \times 10^{-6}.$$

These correspond to  $\theta_a = 0^\circ$ ,  $\theta_b = 45^\circ$ , and  $\theta_c = 90^\circ$  (see figure). Determine the strain components  $\varepsilon_x$ ,  $\varepsilon_y$  and  $\gamma_{xy}$ .

**Solution** For this problem, Eqs. (5.17) become

$$\varepsilon_a = \varepsilon_x, \quad \varepsilon_c = \varepsilon_y, \quad \varepsilon_b = \frac{1}{2}(\varepsilon_x + \varepsilon_y + \gamma_{xy})$$

or

$$\varepsilon_x = \varepsilon_a, \quad \varepsilon_y = \varepsilon_c, \quad \gamma_{xy} = 2\varepsilon_b - (\varepsilon_a + \varepsilon_c).$$

Upon substitution of numerical values,  $\varepsilon_x = 900 \times 10^{-6}$ ,  $\varepsilon_y = -100 \times 10^{-6}$  and  $\gamma_{xy} = 600 \times 10^{-6}$ . The principal stresses and the maximum shearing stresses for these data are found below.

The principal strains can be calculated from the Eq. (5.12):

$$\varepsilon_{1,2} = \left[ \frac{900 - 100}{2} \pm \sqrt{\left( \frac{900 + 100}{2} \right)^2 + \left( \frac{600}{2} \right)^2} \right] \times 10^{-6},$$

from which

$$\varepsilon_1 = 983 \times 10^{-6}, \quad \varepsilon_2 = -183 \times 10^{-6}.$$

As a check, note that  $\varepsilon_x + \varepsilon_y = \varepsilon_1 + \varepsilon_2 = 800 \times 10^{-6}$ .

The maximum shearing strains are given by the Eq. (5.13):

$$\gamma_{\max} = \left[ \pm 2 \sqrt{\left( \frac{900 + 100}{2} \right)^2 + \left( \frac{600}{2} \right)^2} \right] \times 10^{-6} = \pm 1166 \times 10^{-6}.$$

### Example 5.4

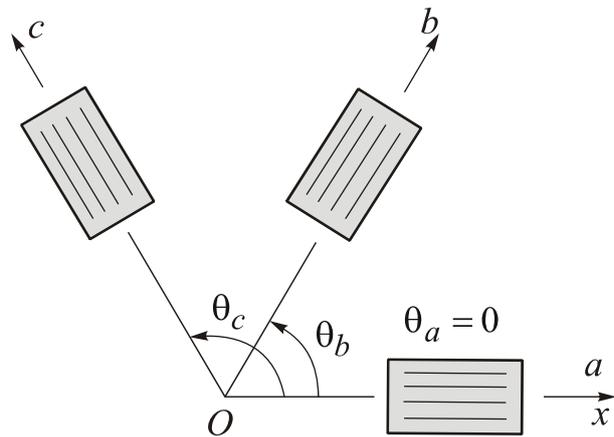
At a point on the free surface of a structure being tested, the  $60^\circ$  rosette readings indicate that

$$\varepsilon_a = 70 \times 10^{-6},$$

$$\varepsilon_b = 850 \times 10^{-6},$$

$$\varepsilon_c = 250 \times 10^{-6}$$

for  $\theta_a = 0^\circ$ ,  $\theta_b = 60^\circ$  and  $\theta_c = 120^\circ$  (see figure). Calculate (1) the in-plane principal strains; (2) in-plane maximum shearing strains and (3) the true maximum shearing strain ( $\nu = 0.3$ ).



(a) Strain rosette

**Solution** In this problem, Eqs. 5.17 reduce to  $\varepsilon_a = \varepsilon_x$ ,

$$\varepsilon_b = \frac{1}{2}(\varepsilon_x + \varepsilon_y) - \frac{1}{4}(\varepsilon_x - \varepsilon_y) + \frac{\sqrt{3}}{4}\gamma_{xy},$$

$$\varepsilon_c = \frac{1}{2}(\varepsilon_x + \varepsilon_y) - \frac{1}{4}(\varepsilon_x - \varepsilon_y) - \frac{\sqrt{3}}{4}\gamma_{xy},$$

which yield

$$\begin{aligned}\varepsilon_x &= \varepsilon_a = 70 \times 10^{-6}, \\ \varepsilon_y &= \frac{1}{3}[2(\varepsilon_b + \varepsilon_c) - \varepsilon_a] = \frac{1}{3}[2(850 \times 10^{-6} + 250 \times 10^{-6}) - 70 \times 10^{-6}] = 710 \times 10^{-6}, \\ \gamma_{xy} &= \frac{2}{\sqrt{3}}(\varepsilon_b - \varepsilon_c) = \frac{2}{\sqrt{3}}(850 \times 10^{-6} - 250 \times 10^{-6}) = 693 \times 10^{-6}.\end{aligned}$$

Equations (5.12) and (5.13) are therefore

$$\begin{aligned}\varepsilon_{1,2} &= \left[ \frac{70 + 710}{2} \pm \sqrt{\left(\frac{70 - 710}{2}\right)^2 + \left(\frac{693}{2}\right)^2} \right] \times 10^{-6}, \\ \gamma_{\max} &= \left[ \pm 2 \sqrt{\left(\frac{70 - 710}{2}\right)^2 + \left(\frac{693}{2}\right)^2} \right] \times 10^{-6},\end{aligned}$$

from which

$$\varepsilon_1 = 862 \times 10^{-6}, \quad \varepsilon_2 = -82 \times 10^{-6}, \quad \gamma_{\max} = \pm 943 \times 10^{-6}.$$

Normal strains corresponding to  $\gamma_{\max}$  are

$$\varepsilon_{\text{aver}} = \frac{1}{2}(70 + 710) = 390 \times 10^{-6}.$$

Orientations of the principal axes are in accordance with Eq. (5.11):

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = \frac{693}{70 - 710} = -1.083.$$

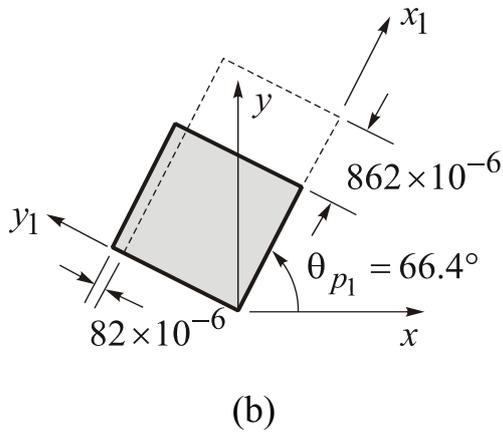
The values of  $2\theta_p$  between  $0$  and  $360^\circ$  are:  $2\theta_p = -47.2^\circ$  and  $2\theta_p = -47.2^\circ + 180^\circ = 132.8^\circ$ , or  $\theta_p = -23.6^\circ$  and  $\theta_p = 66.4^\circ$ .

To determine the value of  $\theta_p$  associated with each principal strain, we substitute  $\theta_p = 66.4^\circ$  into the first transformation equation (Eq. 5.8) and solve for the strain:

$$\begin{aligned}\varepsilon_{x_1} &= \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta = \\ &= (390 \times 10^{-6}) + (320 \times 10^{-6}) (\cos 132.8^\circ) + (346.8 \times 10^{-6}) (\sin 132.8^\circ) = 862 \times 10^{-6}.\end{aligned}$$

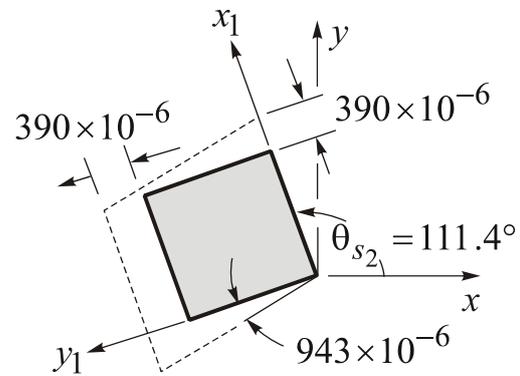
This result shows that the larger principal strain  $\varepsilon_1$  is at the angle  $\theta_{p_1} = 66.4^\circ$  (see figure (b)). Similarly, for  $\theta_s = 66.4^\circ + 45^\circ = 111.4^\circ$ , Eq. (5.9) yields

$\gamma_{x_1y_1} = -\gamma_{\max}$ . This result shows that the element oriented at an angle  $\theta_{s_2} = 111.4^\circ$  has the maximum negative shear strain. The result is given in Fig. c).



(b)

Principal strains and corresponding angles



(c)

Maximum shearing strains and corresponding angles

Using Eq. (5.16) to determine the “out-of-plane” principal strain  $\varepsilon_z$  in terms of the “in-plane” strains  $\varepsilon_x$  and  $\varepsilon_y$ , the “out-of-plane” principal strain is

$$\varepsilon_z = \varepsilon_3 = -\frac{0.3}{1-0.3}(70 + 710) \times 10^{-6} = -334 \times 10^{-6}.$$

The true maximum shearing strain equals

$$(\gamma_{\max})_t = \varepsilon_1 - \varepsilon_3 = (862 + 334) \times 10^{-6} = 1196 \times 10^{-6}.$$

The plane-strain components found can also be used to construct a Mohr's circle.

### Example 5.5

The strain components at a point in a machine part are given by  $\varepsilon_x = 900 \times 10^{-6}$ ,  $\varepsilon_y = -100 \times 10^{-6}$ , and  $\gamma_{xy} = 600 \times 10^{-6}$ . Determine (1) the principal strains and (2) the maximum shearing strains. Show the results on a properly oriented deformed element. The Poisson's ratio  $\nu = 0.3$ .

**Solution** (1) *Principal strains*. The principal strains can be calculated from the Eq. (5.12):

$$\varepsilon_{1,2} = \left[ \frac{900 - 100}{2} \pm \sqrt{\left( \frac{900 + 100}{2} \right)^2 + \left( \frac{600}{2} \right)^2} \right] \times 10^{-6},$$

from which

$$\varepsilon_1 = 983 \times 10^{-6}, \quad \varepsilon_2 = -183 \times 10^{-6}.$$

As a check, note that  $\varepsilon_x + \varepsilon_y = \varepsilon_1 + \varepsilon_2 = 800 \times 10^{-6}$ .

The planes of principal strains and corresponding angles of principal directions are found from Eq. (5.11):

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = \left( \frac{600}{900 + 100} \right) \times 10^{-6} = 0.60.$$

The values of  $2\theta_p$  between 0 and  $360^\circ$  are:

$$2\theta_p = 31^\circ \text{ and } 2\theta_p = 31^\circ + 180^\circ = 211^\circ, \text{ or } \theta_p = 15.5^\circ \text{ and } \theta_p = 105.5^\circ.$$

To determine the value of  $\theta_p$  associated with each principal strain, we substitute  $\theta_p = 15.5^\circ$  into the first transformation equation (Eq. 5.8) and solve for the strain:

$$\begin{aligned} \varepsilon_{x_1} &= \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta = \\ &= (400 \times 10^{-6}) + (500 \times 10^{-6}) (\cos 31^\circ) + (300 \times 10^{-6}) (\sin 31.0^\circ) = 983 \times 10^{-6}. \end{aligned}$$

This result shows that the larger principal strain  $\varepsilon_1$  is at the angle  $\theta_{p_1} = 15.5^\circ$ .

The smaller strain  $\varepsilon_2$  acts at  $90^\circ$  from that direction ( $\theta_{p_2} = 105.5^\circ$ ). Thus,

$$\begin{aligned} \varepsilon_1 &= 983 \times 10^{-6} \text{ and } \theta_{p_1} = 15.5^\circ \left( \varepsilon_1 > \varepsilon_{x_1}, \text{ i.e. } 983 \times 10^{-6} > 900 \times 10^{-6} \right), \\ \varepsilon_2 &= -183 \times 10^{-6} \text{ and } \theta_{p_2} = 105.5^\circ \left( \varepsilon_2 < \varepsilon_{y_1}, \text{ i.e. } (-183 \times 10^{-6}) < (-100 \times 10^{-6}) \right). \end{aligned}$$

The principal strains are shown in Fig. a.

**Note, that there are no shear strains on the principal plains.**

(2) *Maximum shearing strains.* They are given by the Eq. (5.13):

$$\frac{\gamma_{\max}}{2} = \left[ \pm \sqrt{\left( \frac{900 + 100}{2} \right)^2 + \left( \frac{600}{2} \right)^2} \right] \times 10^{-6} = \pm 583 \times 10^{-6}.$$

The algebraically maximum shear strain  $\gamma_{\max}$  equals to  $+1166 \times 10^{-6}$ . The algebraically minimum shear strain has the same magnitude, but is negative.

The element having the maximum shear strains is oriented at  $45^\circ$  to the principal directions; therefore,  $\theta_s = 15.5^\circ + 45^\circ = 60.5^\circ$  and  $2\theta_s = 121.0^\circ$ . By substituting this value of  $2\theta_s$  into the second transformation equation (Eq. 5.9), we can determine the sign of the shear strain associated with this direction. The calculations are as follows:

$$\frac{\gamma_{x_1y_1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2} \sin 2\theta + \frac{\gamma_{xy}}{2} \cos 2\theta =$$

$$= -(500 \times 10^{-6}) (\sin 121.0^\circ) + (300 \times 10^{-6}) (\cos 121.0^\circ) = -583 \times 10^{-6}.$$

This result shows that an element oriented at an angle  $\theta_{s_2} = 60.5^\circ$  has the maximum negative shear strain.

We can get the same result by observing that the angle  $\theta_{s_1}$  to the direction of maximum positive shear strain is always  $45^\circ$  less than  $\theta_{p_1}$ . Hence,

$$\theta_{s_1} = \theta_{p_1} - 45^\circ = 15.5^\circ - 45^\circ = -29.5^\circ,$$

$$\theta_{s_2} = \theta_{s_1} + 90^\circ = 60.5^\circ.$$

The shear strains corresponding to  $\theta_{s_1}$  and  $\theta_{s_2}$  are  $\gamma_{\max} = +1166 \times 10^{-6}$  and  $\gamma_{\min} = -1166 \times 10^{-6}$ , respectively. A sketch of the element having the maximum in-plane shear strains is shown in Fig. b. The initial and deformed elements are indicated by the solid and dashed lines, respectively.

The normal strains associated with the axes of  $\gamma_{\max}$  are represented by the formula

$$\varepsilon_{\text{aver}} = \frac{\varepsilon_x + \varepsilon_y}{2} = \left( \frac{900 - 100}{2} \right) \times 10^{-6} = 400 \times 10^{-6}.$$

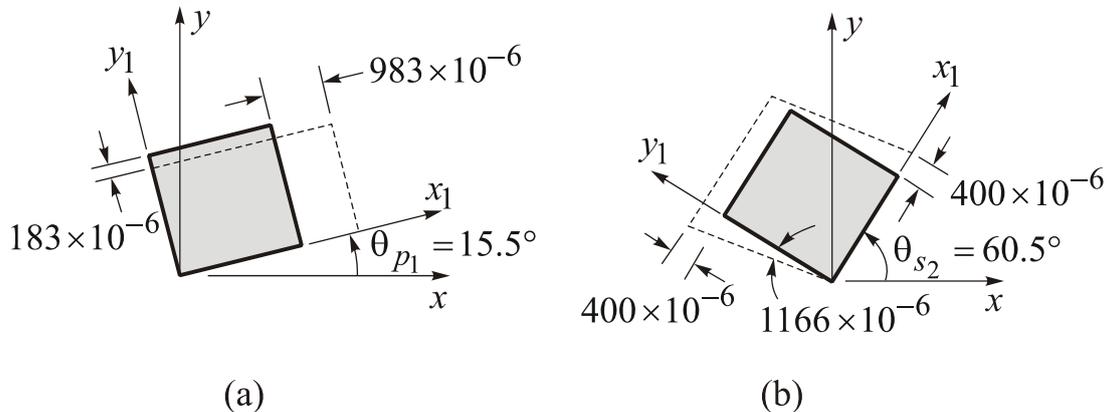
If a state of plane stress exists in the member under consideration, the third principal strain for  $\nu$  equals to 0.3 is from Eq. (5.16):

$$\varepsilon_z = \varepsilon_3 = -\frac{0.3}{1 - 0.3} (900 - 100) \times 10^{-6} = -343 \times 10^{-6}.$$

Then the true maximum shear strain equals

$$(\gamma_{\max})_t = \varepsilon_1 - \varepsilon_3 = (983 + 343) \times 10^{-6} = 1326 \times 10^{-6}.$$

**Note. In this case, the maximum in-plane shear strain does not represent the largest shearing strain.**



Element of material in plane strain: (a) principal strains; (b) maximum shear strains



## PROBLEMS

**Problem 5.1 through 5.5** The state of strain at specific points is given in the table below. Determine the state of strain associated with the specified angle  $\theta$ . Use Eqs. (5.8, 5.9).

Problem	$\varepsilon_x,$ $10^{-6}$	$\varepsilon_y,$ $10^{-6}$	$\gamma_{xy},$ $10^{-6}$	$\theta$
5.1	-500	600	1500	$30^\circ$
5.2	500	1200	-1200	$-45^\circ$
5.3	200	400	100	$45^\circ$
5.4	-300	500	200	$60^\circ$
5.5	-750	-200	200	$-30^\circ$

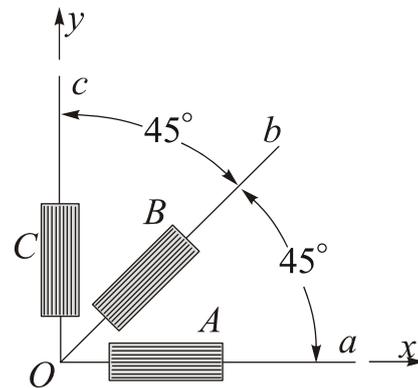
**Problem 5.6 through 5.15** The state of strain at a point in a thin plate in the table below is given. Calculate (1) the in-plane principal strains and the maximum in-plane shear strain and (2) the true maximum shearing strain ( $\nu = 0.3$ ). Sketch the results on properly oriented deformed elements.

Problem	$\varepsilon_x,$ $10^{-6}$	$\varepsilon_y,$ $10^{-6}$	$\gamma_{xy},$ $10^{-6}$
5.6	400	0	300
5.7	-900	-400	-300
5.8	-750	100	500
5.9	-200	600	600
5.10	520	100	150
5.11	830	-300	250
5.12	-360	-500	-300
5.13	-500	500	-500
5.14	-300	800	650
5.15	850	350	100

**Problem 5.16** During the static test of an aircraft panel, a  $45^\circ$  rosette measures the following normal strains on the free surface:

$$\varepsilon_a = -400 \times 10^{-6}, \varepsilon_b = -500 \times 10^{-6},$$

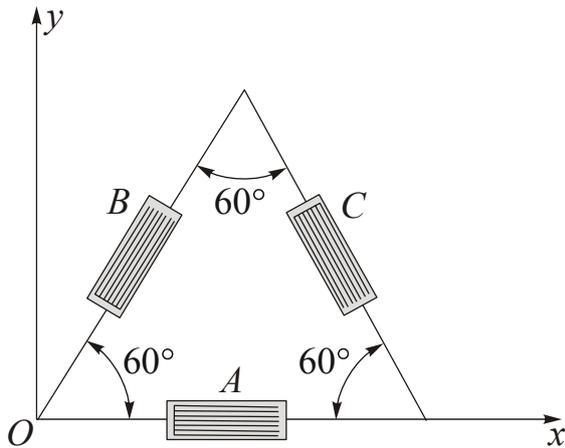
$\varepsilon_c = 200 \times 10^{-6}$ . Calculate the principal strains. Show the results on a properly oriented deformed element.



**Problem 5.17** Using a  $60^\circ$  rosette, we find the following strains at a critical point on the frame of a stressed beam:

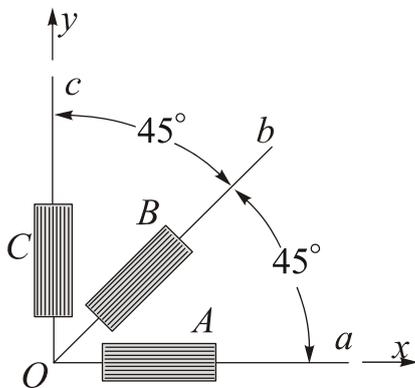
$$\varepsilon_a = -200 \times 10^{-6}, \varepsilon_b = -350 \times 10^{-6},$$

and  $\varepsilon_c = -550 \times 10^{-6}$ . Determine (1) the maximum in-plane shear strains and the accompanying normal strains and (2) the true maximum shear strain. Use  $\nu = 0.3$ . Sketch the results on a properly oriented distorted element.



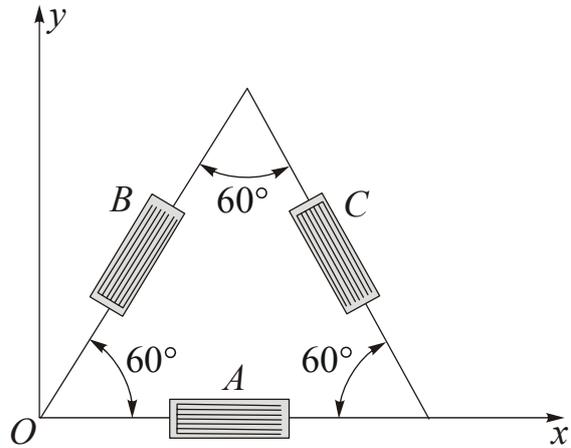
**Problem 5.18** Verify that, for a 45° rosette, the principal strains are expressed as follows:

$$\varepsilon_{1,2} = \frac{\varepsilon_a + \varepsilon_c}{2} \pm \frac{1}{2} \left[ 2\varepsilon_a(\varepsilon_a - 2\varepsilon_b) + 2\varepsilon_c(\varepsilon_c - 2\varepsilon_b) + 4\varepsilon_b^2 \right]^{\frac{1}{2}}$$

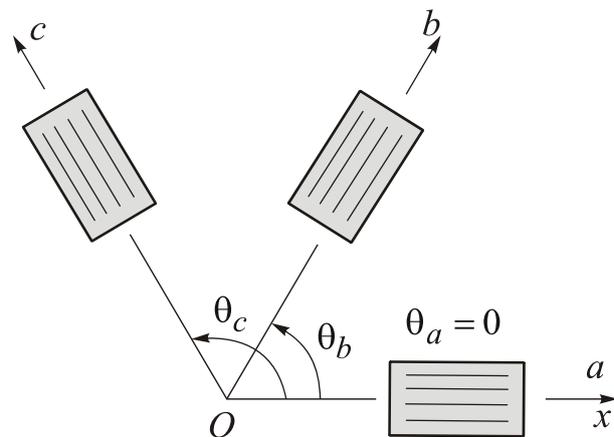


**Problem 5.19** At a point on the free surface of a steel member ( $E = 200$  GPa,  $\nu = 0.3$ ) subjected to plane stress, a 60° rosette measures the strains  $\varepsilon_a = 1200 \times 10^{-6}$ ,  $\varepsilon_b = -650 \times 10^{-6}$  and  $\varepsilon_c = 550 \times 10^{-6}$ . Determine (1) the principal strains and their directions and (2) the corresponding principal stresses and the maximum shear stress. Sketch

the results on a properly oriented element.

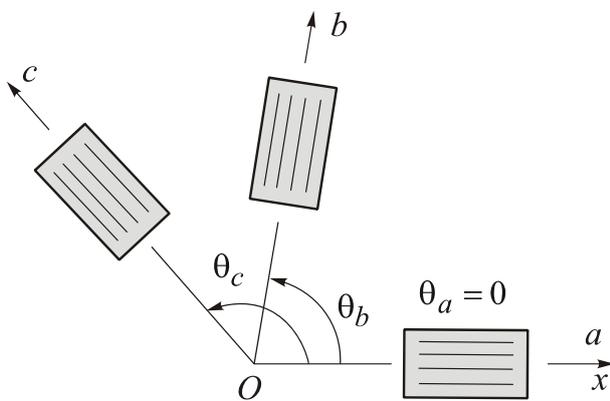


**Problem 5.20** At a point on the surface of a stressed structure (see figure), the strains are  $\varepsilon_a = -200 \times 10^{-6}$ ,  $\varepsilon_b = -500 \times 10^{-6}$ , and  $\varepsilon_c = -900 \times 10^{-6}$  for  $\theta_a = 0^\circ$ ,  $\theta_b = 120^\circ$ , and  $\theta_c = 240^\circ$ . Calculate (1) the in-plane principal strains and (2) the in-plane maximum shear strains. Show the results on properly oriented deformed elements.

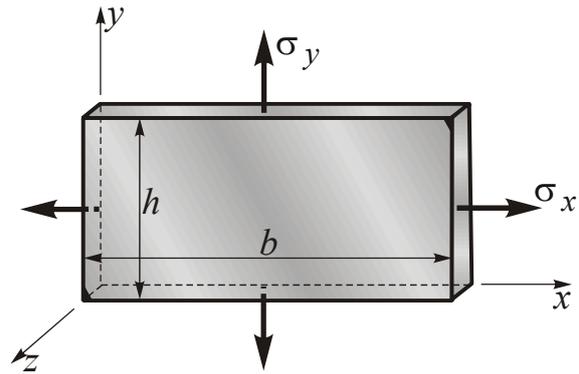


**Problem 5.21** The strain readings at a point on the free surface of a member subjected to plane stress (see

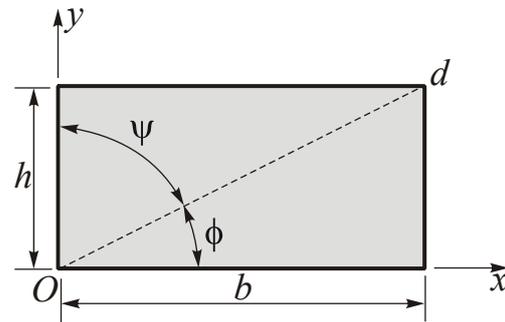
figure) are  $\epsilon_a = 500 \times 10^{-6}$ ,  $\epsilon_b = 350 \times 10^{-6}$ , and  $\epsilon_c = 800 \times 10^{-6}$  for  $\theta_a = 0^\circ$ ,  $\theta_b = 60^\circ$ , and  $\theta_c = 135^\circ$ . Calculate (1) the maximum in-plane shearing strains and (2) the true maximum shear strain ( $\nu = 0.33$ ). Sketch the results on a properly oriented distorted element.



**Problem 5.22** A thin rectangular plate in biaxial stress is subjected to stresses  $\sigma_x$  and  $\sigma_y$ , as shown in figure (a). The width and height of the plate are  $b = 20$  cm and  $h = 10$  cm, respectively. Measurements show that the normal strains in the  $x$  and  $y$  directions are  $\epsilon_x = 195 \times 10^{-6}$  and  $\epsilon_y = -125 \times 10^{-6}$ , respectively. With reference to figure (b), determine the following quantities: (1) the increase  $\Delta d$  in the length of diagonal  $Od$ ; (2) the change  $\Delta\phi$  in the angle  $\phi$  between diagonal  $Od$  and the  $x$  axis; and (3) the change  $\Delta\psi$  in the angle  $\psi$  between diagonal  $Od$  and the  $y$  axis.

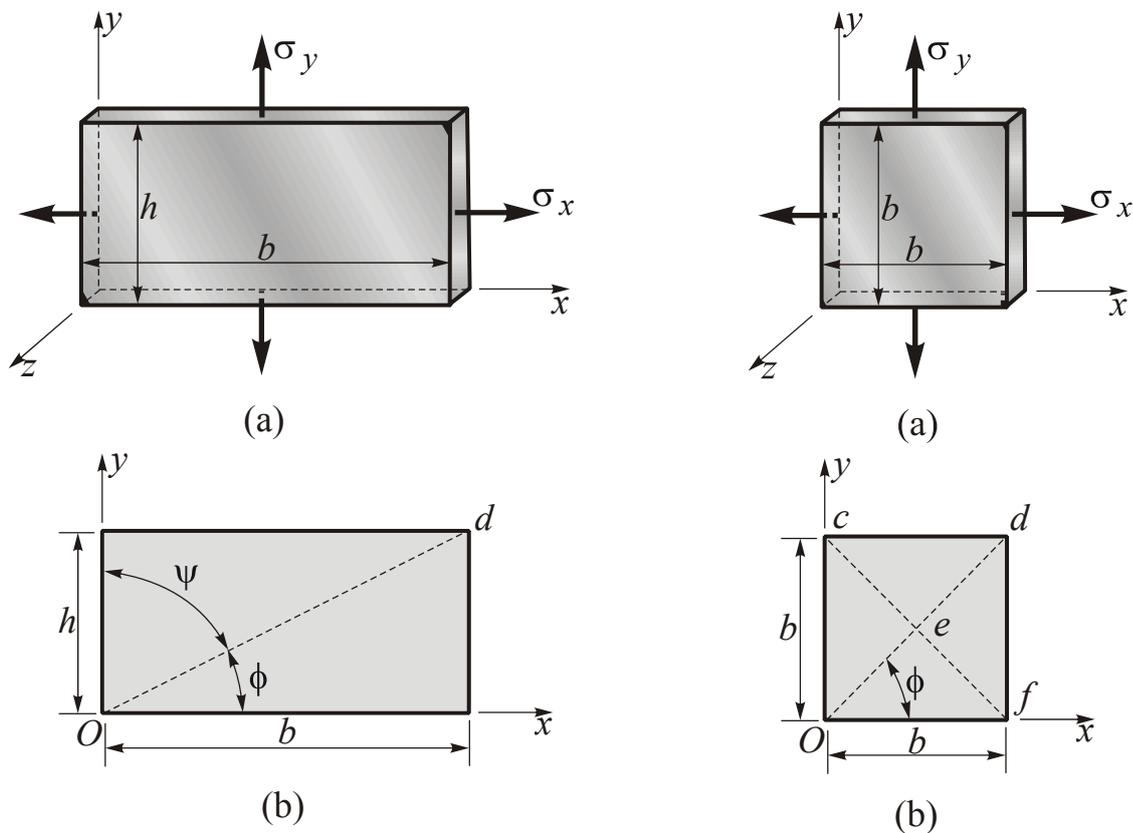


(a)



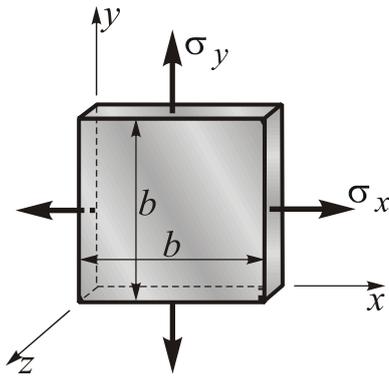
(b)

**Problem 5.23** A thin rectangular plate in biaxial stress is subjected to stresses  $\sigma_x$  and  $\sigma_y$ , as shown in figure (a). The width and height of the plate are  $b = 140$  mm and  $h = 90$  mm, respectively. Measurements show that the normal strains in the  $x$  and  $y$  directions are  $\epsilon_x = 102 \times 10^{-6}$  and  $\epsilon_y = -31 \times 10^{-6}$ , respectively. Using two-dimensional view of the plate (see figure (b)), determine the following quantities: (1) the increase  $\Delta d$  in the length of diagonal  $Od$ ; (2) the change  $\Delta\phi$  in the angle  $\phi$  between diagonal  $Od$  and the  $x$  axis; and (3) the change  $\Delta\psi$  in the angle  $\psi$  between diagonal  $Od$  and the  $y$  axis.

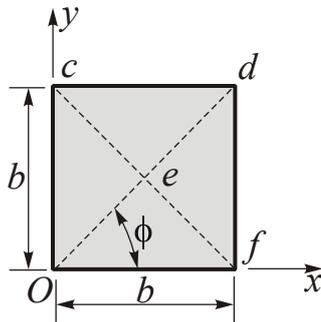


**Problem 5.24** A thin square plate in biaxial stress is subjected to stresses  $\sigma_x$  and  $\sigma_y$ , as shown in figure (a). The width of the plate is  $b = 30$  mm. Measurements show that the normal strains in the  $x$  and  $y$  directions are  $\varepsilon_x = 214 \times 10^{-6}$  and  $\varepsilon_y = 56 \times 10^{-6}$ , respectively. Determine the following quantities (see figure (b)): (1) the increase  $\Delta d$  in the length of diagonal  $Od$ ; (2) the change  $\Delta\phi$  in the angle  $\phi$  between diagonal  $Od$  and the  $x$  axis; and (3) the shear strain  $\gamma$  associated with diagonals  $Od$  and  $cf$  (that is, find the decrease in angle  $ced$ ).

**Problem 5.25** A thin square plate in biaxial stress is subjected to stresses  $\sigma_x$  and  $\sigma_y$ , as shown in figure (a). The width of the plate is  $b = 250$  mm. Measurements show that the normal strains in the  $x$  and  $y$  directions are  $\varepsilon_x = 427 \times 10^{-6}$  and  $\varepsilon_y = 113 \times 10^{-6}$ , respectively. Determine the following quantities: (1) the increase  $\Delta d$  in the length of diagonal  $Od$ ; (2) the change  $\Delta\phi$  in the angle  $\phi$  between diagonal  $Od$  and the  $x$  axis; and (3) the shear strain  $\gamma$  associated with diagonals  $Od$  and  $cf$  (that is, find the decrease in angle  $ced$ ).

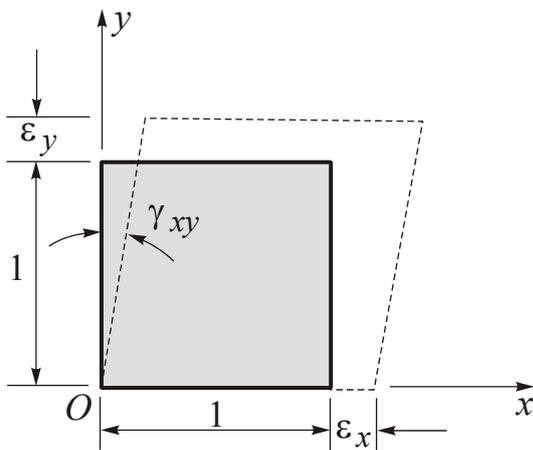


(a)

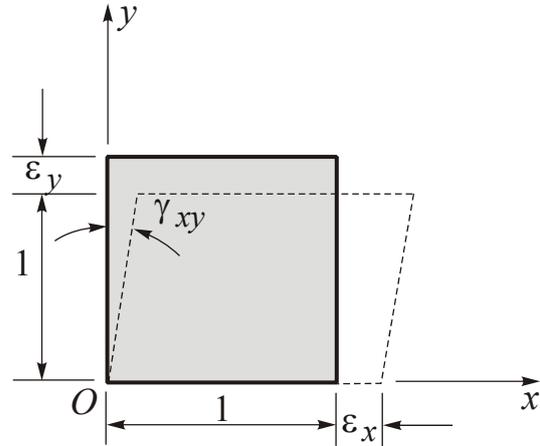


(b)

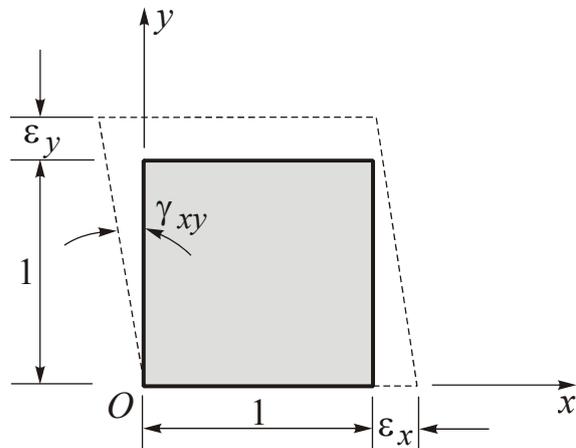
**Problem 5.26** An element subjected to plane strain (see figure) has strains as follows:  $\epsilon_x = 110 \times 10^{-6}$ ,  $\epsilon_y = 240 \times 10^{-6}$  and  $\gamma_{xy} = 90 \times 10^{-6}$ . Calculate the strains for an element oriented at an angle  $\theta = 30^\circ$  and show these strains on a sketch of a properly oriented element.



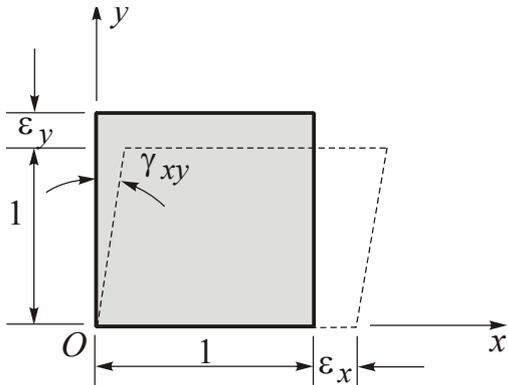
**Problem 5.27** An element of material subjected to plane strain (see figure) has strains as follows:  $\epsilon_x = 420 \times 10^{-6}$ ,  $\epsilon_y = -180 \times 10^{-6}$  and  $\gamma_{xy} = 300 \times 10^{-6}$ . Calculate the strains for an element oriented at an angle  $\theta = 50^\circ$  and show these strains on a sketch of a properly oriented element.



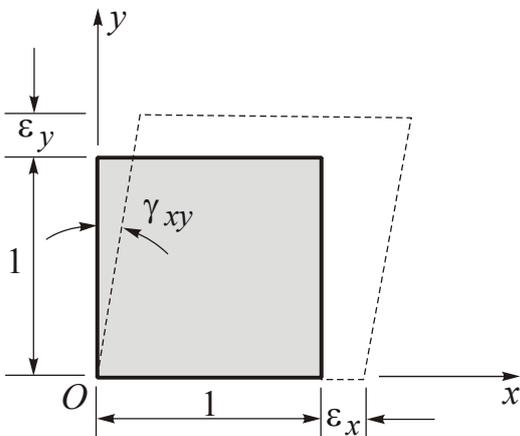
**Problem 5.28** The strains in plane strain (see figure) are as follows:  $\epsilon_x = 500 \times 10^{-6}$ ,  $\epsilon_y = 150 \times 10^{-6}$ , and  $\gamma_{xy} = -340 \times 10^{-6}$ . Determine the principal strains and maximum shear strains, and show these strains on sketches of properly oriented elements.



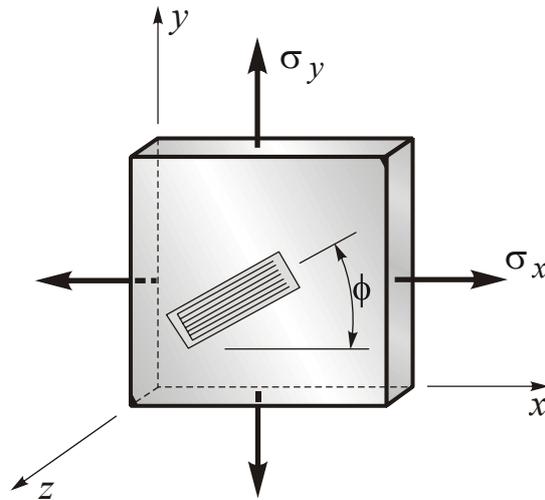
**Problem 5.29** The strains for an element in plane strain (see figure) are as follows:  $\epsilon_x = 120 \times 10^{-6}$ ,  $\epsilon_y = -450 \times 10^{-6}$ , and  $\gamma_{xy} = 360 \times 10^{-6}$ . Determine the principal strains and maximum shear strains, and show these strains on sketches of properly oriented elements.



**Problem 5.30** An element of material in plane strain (see figure) is subjected to strains  $\epsilon_x = 480 \times 10^{-6}$ ,  $\epsilon_y = 70 \times 10^{-6}$ , and  $\gamma_{xy} = 420 \times 10^{-6}$ . Determine the following quantities: (1) the strains for an element oriented at an angle  $\theta = 70^\circ$ ; (2) the principal strains, and (3) the maximum shear strains. Show the results on sketches of properly oriented elements.

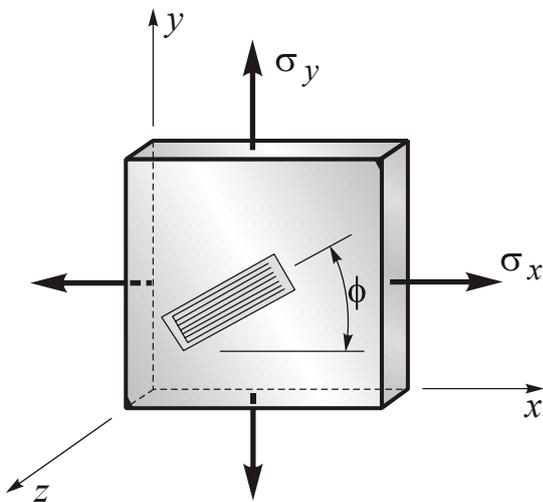


**Problem 5.31** A steel plate with modulus of elasticity  $E = 210 \times 10^6$  GPa and Poisson's ratio  $\nu = 0.30$  is loaded in biaxial stress by normal stresses  $\sigma_x$  and  $\sigma_y$  (see figure). A strain gage is bonded to the plate at an angle  $\phi = 30^\circ$ . If the stress  $\sigma_x$  is 124 MPa and the strain measured by the gage is  $\epsilon = 407 \times 10^{-6}$ , what is the maximum in-plane shear stress  $(\tau_{\max})_{xy}$  and shear strain  $(\gamma_{\max})_{xy}$ ? What is the maximum shear strain  $(\gamma_{\max})_{xz}$  in the  $xz$  plane? What is the maximum shear strain  $(\gamma_{\max})_{yz}$  in the  $yz$  plane?

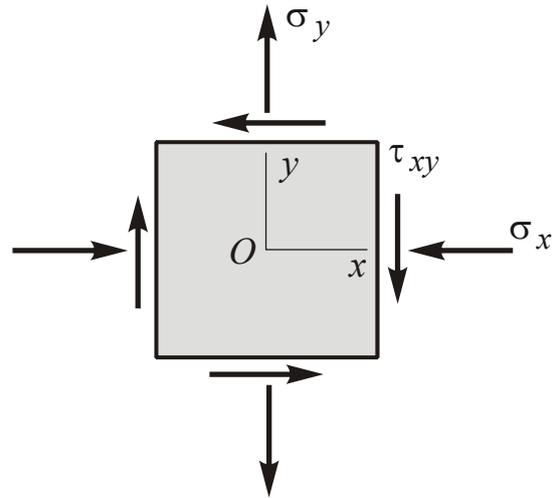


**Problem 5.32** An aluminum plate with modulus of elasticity  $E = 70$  GPa and Poisson's ratio  $\nu = 0.30$  is loaded in biaxial stress by normal stresses  $\sigma_x$  and  $\sigma_y$  (see figure). A strain gage is bonded to the plate at an angle  $\phi = 21^\circ$ . If the stress  $\sigma_x$  is 86.4 MPa and the strain measured by

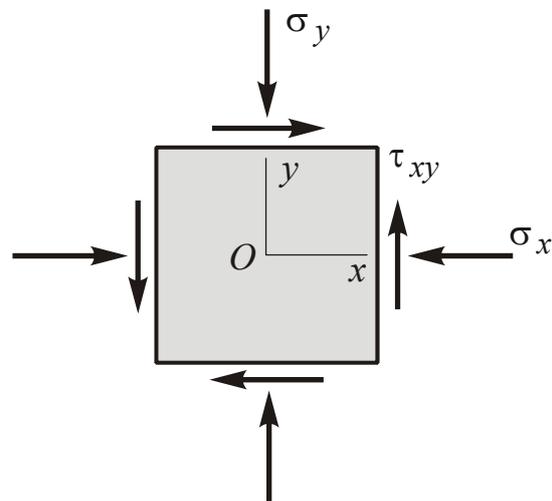
the gage is  $\epsilon = 946 \times 10^{-6}$ , what is the maximum in-plane shear stress  $(\tau_{\max})_{xy}$  and shear strain  $(\gamma_{\max})_{xy}$ ? What is the maximum shear strain  $(\gamma_{\max})_{xz}$  in the  $xz$  plane? What is the maximum shear strain  $(\gamma_{\max})_{yz}$  in the  $yz$  plane?



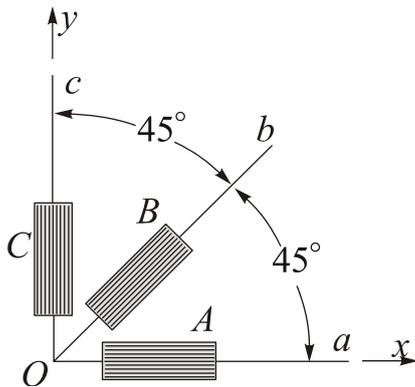
**Problem 5.33** An aluminum element in plane stress is subjected to stresses  $\sigma_x = -58$  MPa,  $\sigma_y = 7.6$  MPa, and  $\tau_{xy} = -10$  MPa (see figure). The modulus of elasticity  $E = 70$  GPa and Poisson's ratio  $\nu = 0.33$ . Determine the following quantities: (1) the strains for an element oriented at an angle  $\theta = 35^\circ$ ; (2) the principal strains, and (3) the maximum shear strains. Show the results on sketches of properly oriented elements.



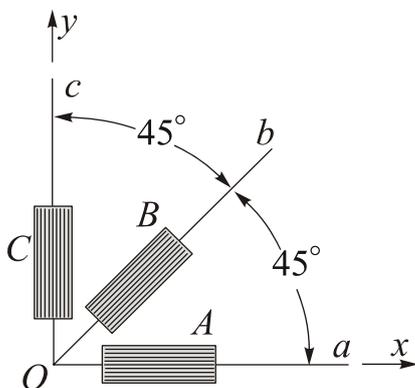
**Problem 5.34** An element of brass in plane stress is subjected to stresses  $\sigma_x = -150$  MPa,  $\sigma_y = -210$  MPa, and  $\tau_{xy} = 15$  MPa (see figure). The modulus of elasticity  $E = 100$  GPa and Poisson's ratio  $\nu = 0.34$ . Determine the following quantities: (1) the strains for an element oriented at an angle  $\theta = 50^\circ$ ; (2) the principal strains, and (3) the maximum shear strains. Show the results on sketches of properly oriented elements.



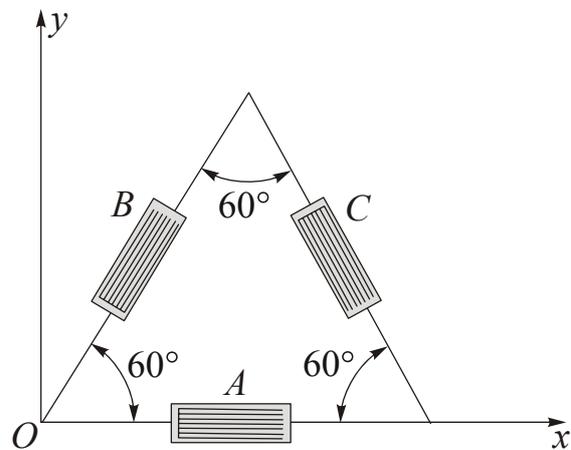
**Problem 5.35** During a test of an airplane wing, the strain gage readings from a  $45^\circ$  rosette (see figure) are as follows: gage  $A$ ,  $520 \times 10^{-6}$ ; gage  $B$ ,  $360 \times 10^{-6}$ ; and gage  $C$ ,  $-80 \times 10^{-6}$ . Determine the principal strains and maximum shear strains, and show them on sketches of properly oriented elements.



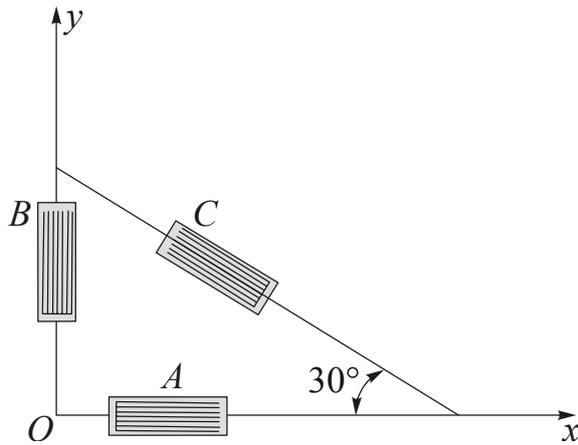
**Problem 5.36** A  $45^\circ$  strain rosette (see figure) mounted on the surface of a frame gives the following readings: gage  $A$ ,  $\epsilon_a = 310 \times 10^{-6}$ ; gage  $B$ ,  $\epsilon_b = 280 \times 10^{-6}$ ; and gage  $C$ ,  $\epsilon_c = -160 \times 10^{-6}$ . Determine the principal strains and maximum shear strains, and show them on sketches of properly oriented elements.



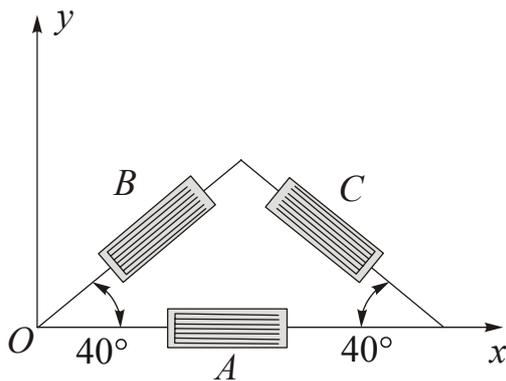
**Problem 5.37** A  $60^\circ$  strain rosette is shown in the figure. Gage  $A$  measures the normal strain  $\epsilon_a$  in the direction of the  $x$  axis. Gages  $B$  and  $C$  measure the strains  $\epsilon_b$  and  $\epsilon_c$  in the inclined directions shown. Obtain the equations for the strains  $\epsilon_x$ ,  $\epsilon_y$ , and  $\gamma_{xy}$  associated with the  $xy$  axes.



**Problem 5.38** On the surface of a structural element in a space vehicle, the strains are monitored by means of three strain gages arranged as shown in the figure. During a flight, the following strains were recorded:  $\epsilon_a = 1100 \times 10^{-6}$ ,  $\epsilon_b = 200 \times 10^{-6}$ , and  $\epsilon_c = 200 \times 10^{-6}$ . Determine the principal strains and principal stresses in the material, which is a magnesium alloy for which  $E = 41 \text{ GPa}$  and  $\nu = 0.35$ . Show the principal strains and principal stresses on sketches of properly oriented elements.

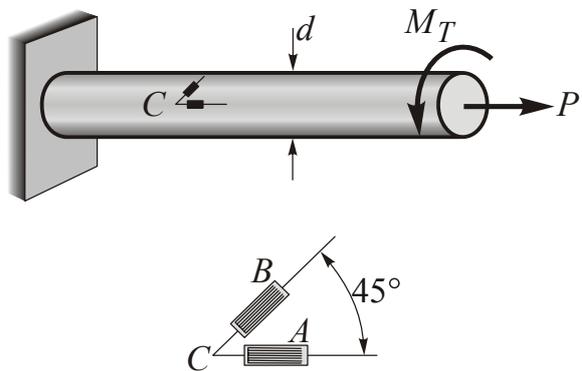


**Problem 5.39** The strains on the surface of structural element made of aluminum alloy ( $E = 70 \text{ GPa}$ ,  $\nu = 0.33$ ) and tested in a space shuttle were measured by means of strain gages. The gages were oriented as shown in the figure, and the measured strains were  $\epsilon_a = 1100 \times 10^{-6}$ ,  $\epsilon_b = 1496 \times 10^{-6}$ , and  $\epsilon_c = -39.44 \times 10^{-6}$ . What is the stress  $\sigma_x$  in the  $x$  direction?

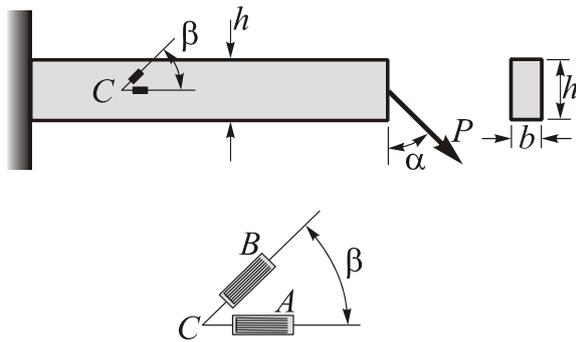


**Problem 5.40** A solid circular bar of diameter  $d = 3 \text{ cm}$  is subjected to an axial force  $P$  and a torque moment  $M_t$  (see figure). Strain gages  $A$  and  $B$  mounted on the surface of the bar give

readings  $\epsilon_a = 100 \times 10^{-6}$  and  $\epsilon_b = -55 \times 10^{-6}$ . The bar is made of steel having  $E = 210 \text{ GPa}$  and  $\nu = 0.30$ . (1) Determine the axial force  $P$  and the torque moment  $M_t$ . (2) Determine the maximum shear strain  $\gamma_{\max}$  and the maximum shear stress  $\tau_{\max}$  in the bar.

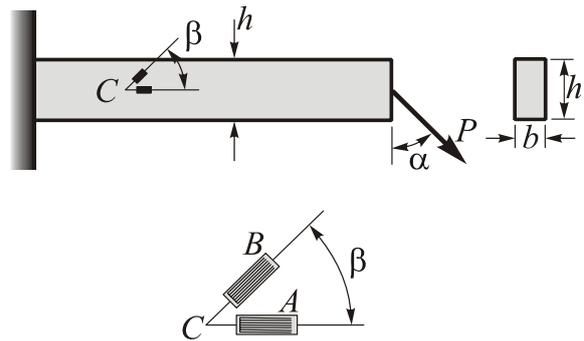


**Problem 5.41** A cantilever beam of rectangular cross section (width  $b = 25 \text{ mm}$ , height  $h = 100 \text{ mm}$ ) is loaded by a force  $P$  that acts at the midheight of the beam and is inclined at an angle  $\alpha$  to the vertical (see figure). Two strain gages are placed at point  $C$ , which also is at the midheight of the beam. Gage  $A$  measures the strain in the horizontal direction and gage  $B$  measures the strain at an angle  $\beta = 60^\circ$  to the horizontal. The measured strains are  $\epsilon_a = 120 \times 10^{-6}$  and  $\epsilon_b = -370 \times 10^{-6}$ . Determine the force  $P$  and the angle  $\alpha$ , assuming the material is steel with  $E = 200 \text{ GPa}$  and  $\nu = 0.33$ .



midheight of the beam. Gage *A* measures the strain in the horizontal direction and gage *B* measures the strain at an angle  $\beta = 75^\circ$  to the horizontal. The measured strains are  $\varepsilon_a = 171 \times 10^{-6}$  and  $\varepsilon_b = -262 \times 10^{-6}$ . Determine the force  $P$  and the angle  $\alpha$ .

**Problem 5.42** A cantilever beam of rectangular cross section (width  $b = 20$  mm, height  $h = 80$  mm, magnesium alloy with  $E = 41$  GPa and  $\nu = 0.35$ ) is loaded by a force  $P$  that acts at the midheight of the beam and is inclined at an angle  $\alpha$  to the vertical (see figure). Two strain gages are placed at point *C*, which also is at the



## Chapter 6 Limiting Stress State. Uniaxial Limiting Stress State. Yield and Fracture Criteria for Combined Stress

The mechanical behavior of materials subject to uniaxial loads is presented on stress-strain diagrams. In these cases, the onset of inelastic behavior, or failure by yielding or brittle fracture can *be predicted readily with acceptable accuracy*. However, most structures are subjected to a variety of combined loads. Several *theories of failure* (syn. *failure criteria*) have been developed for predicting failure of brittle and ductile materials in these situations, when in the point under consideration two or three principal stresses are applied. A *yield criterion*, often expressed as *yield surface*, or *yield locus*, is an hypothesis concerning the limit of elasticity under any combination of stresses. Since stress and strain are tensor qualities they can be described on the basis of three principal directions, in the case of stress these are denoted by  $\sigma_1$ ,  $\sigma_2$ , and  $\sigma_3$ .

We consider here only two simple brittle fracture and three yield theories (criteria). In our discussion of these theories, we denote the ultimate stress obtained in a tension test by  $\sigma_{ult_t}$  and in a compression test by  $\sigma_{ult_c}$ . The yield-point stress determined from a tensile test is designated by  $\sigma_y$ . In this analysis, we will consider an element subjected to triaxial principal stresses where  $\sigma_1 > \sigma_2 > \sigma_3$  and the subscripts 1, 2, and 3 refer to the principal directions. The following represents these most common criteria as applied to anisotropic materials (uniform properties in all directions).

### 6.1 Maximum Principal Stress Theory (Rankine, Lamé)

The maximum principal stress theory, or *Rankine theory*, is applied satisfactorily to many brittle materials. It is based on a limiting normal stress and states that *fracture occurs when either of the principal stress  $\sigma_1$  or  $\sigma_3$  at a point in the structure reaches the ultimate stress in simple tension or compression* for the material. It follows that fracture impends when

$$\sigma_1 = \sigma_{ult_t} \quad \text{or} \quad |\sigma_3| = \sigma_{ult_c}. \quad (6.1)$$

For materials possessing the same ultimate strength in tension and compression ( $\sigma_{ult_t} = \sigma_{ult_c} = \sigma_u$ ), in the case of *plane stress* ( $\sigma_3 = 0$ ), Eq. (6.1) becomes

$$\sigma_1 = \sigma_u \quad \text{or} \quad |\sigma_2| = \sigma_u. \quad (6.2)$$

Failure is prevented when

$$\sigma_1 < \sigma_u \quad \text{or} \quad |\sigma_2| < \sigma_u. \quad (6.3)$$

The above expression is plotted in Fig. 6.1. Failure will occur for any combination of stresses on or outside the boundaries; *no fracture occurs for a combination of stresses inside the square.*

Experiments show that this theory can predict fracture failures reasonably well for brittle materials (particularly in quadrant 1 in Fig. 6.1), and the Rankine theory is generally accepted in design practice for such materials.

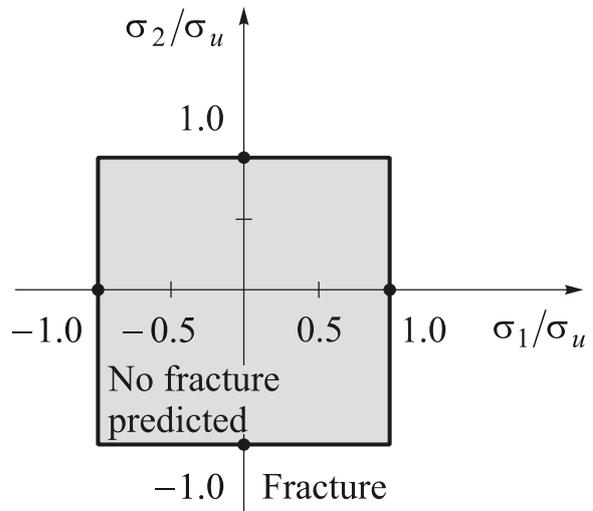


Fig. 6.1 Fracture criterion based on maximum principal stress

### 6.2 Maximum Principal Strain Theory (Saint-Venant)

The theory is based on the assumption that inelastic behavior or failure is governed by a specified maximum normal strain. *Failure will occur* at a particular part in a body subjected to an arbitrary state of strain *when the normal strain reaches a limiting level corresponding to the yield point during a simple tensile test.* Failure is predicted when either of the principal strains, resulting from the principal stresses,  $\sigma_{1,2}$ , equals or exceeds the maximum strain corresponding to the yield strength,  $\sigma_y$ , of the material in uniaxial tension or compression. In terms of the principal stresses the fracture will be prevented when

$$\sigma_1 - \nu(\sigma_2 + \sigma_3) < \sigma_y \tag{6.4}$$

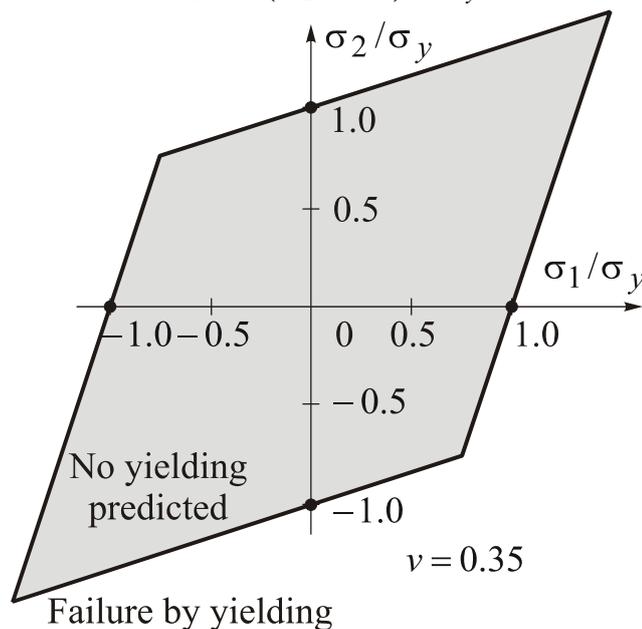


Fig. 6.2 Fracture criterion based on maximum principal strain

Equation (6.4) is depicted in Fig. 6.2. The boundaries of the parallelogram mark the onset yielding, with points outside the shaded region representing the yielded state.

### 6.3 Maximum Shear Stress Theory (Tresca, Guest, Coulomb)

Applied satisfactory to ductile materials, the theory is based on the concept on limiting shearing stresses at which failure occurs. It asserts that *yielding begins when the maximum shearing stress equals the maximum shearing stress at the yield point in a simple tension test*. The largest value of the shear stress is

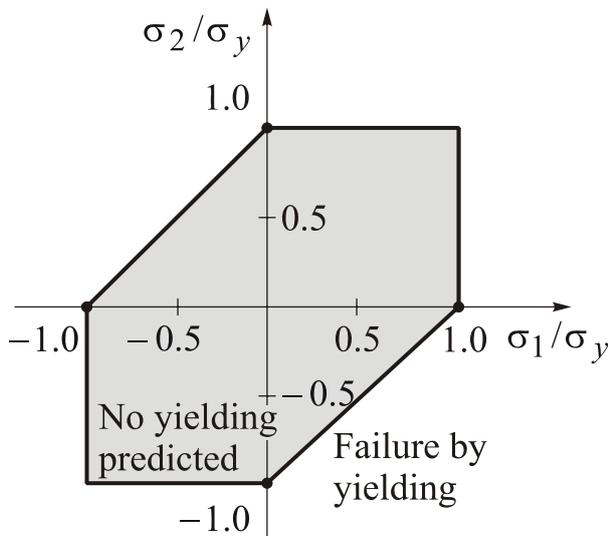
$$\tau_{\max} = \frac{1}{2}|\sigma_1 - \sigma_3|. \text{ In uniaxial limiting tension, } \sigma_2 = \sigma_3 = 0 \text{ and } \tau_{\max} = \sigma_y/2.$$

Therefore, at the onset of yielding,

$$\tau_{\max} = \sigma_y \text{ or } |\sigma_1 - \sigma_3| = \sigma_y. \tag{6.5}$$

In the case of *plane stress* ( $\sigma_3 = 0$ ), when  $\sigma_1$  and  $\sigma_2$  are of *opposite sign* (that is, one tensile, the other compressive), the yield condition is given by

$$|\sigma_1 - \sigma_2| = \sigma_y. \tag{6.6}$$



When  $\sigma_1$ , and  $\sigma_2$  carry the same sign, the maximum shearing stress is half the numerically larger principal stress  $\sigma_1$  or  $\sigma_2$ . Thus, the criterion corresponding to this situation is

$$|\sigma_1| = \sigma_y \text{ or } |\sigma_2| = \sigma_y. \tag{6.7}$$

Equations (6.6) and (6.7) are depicted in Fig. 6.3. Note that Eq. (6.6) applies to the second and fourth quadrants. In the first and third quadrants the criteria are expressed by Eq. (6.7). The boundaries of the hexagon mark the onset yielding, with points outside the shaded region representing the yielded state.

Fig. 6.3 Yield criterion based on maximum shearing stress

The maximum shear stress theory is frequently applied in machine design because it is slightly conservative and is easy to apply. Good agreement nth experiment has been realized for ductile materials. The main objection of this theory is that it ignores the possible effect of the intermediate principal stress,  $\sigma_2$ .

However, only one other theory, *the maximum distortional strain energy theory*, predicts yielding better than does the Tresca theory, and the differences between the two theories are rarely more than 15%.

### 6.4 Total Strain Energy Theory (Beltrami-Haigh)

Applicable to many types of materials, the *theory predicts failure* or inelastic behavior at a point *when the total strain energy per unit volume associated with the principal stresses,  $\sigma_{1,2,3}$ , equals or exceeds the total strain energy corresponding to that for the yield strength,  $\sigma_y$ , of the material in uniaxial tension or compression.* Failure is prevented when

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3) < \sigma_y^2. \quad (6.8)$$

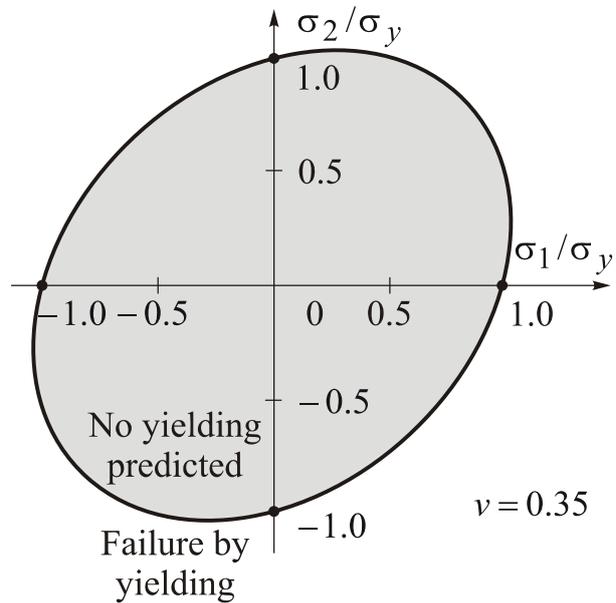


Fig. 6.4 Yield criterion based on maximum total strain energy

### 6.5 Maximum Distortion Energy Theory (Huber-Henky-von Mises)

The theory is based on a limiting *energy of distortion*, i.e. energy associated with shear strains.

Strain energy can be separated into energy associated with volume change and energy associated with distortion of the body. *The maximum distortion energy failure theory assumes failure by yielding* in a more complicated loading situation to occur *when the distortion energy in the material reaches the same value as in a tension test at yield.*

This theory provides the best agreement between experiment and theory and, along the Tresca theory, is very widely used today.

Failure is predicted when the distortional energy associated with the principal stresses,  $\sigma_{1,2,3}$ , equals or exceeds the distortional energy corresponding to that for the yield strength,  $\sigma_y$ , of the material in uniaxial tension or compression. The failure is prevented when

$$\left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] < 2\sigma_y^2. \quad (6.9)$$

In the case of *plane stress* ( $\sigma_3 = 0$ ), Eq. (6.9) reduces to

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 < \sigma_y^2. \quad (6.10)$$

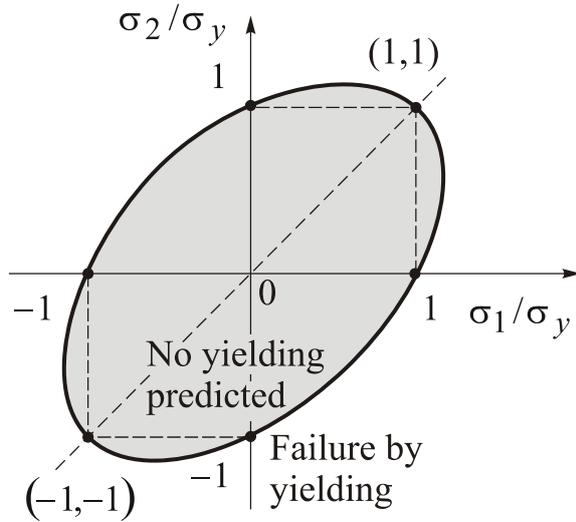


Fig. 6.5 Yield criterion based on maximum distortion energy

The foregoing defines the ellipse shown in Fig.6.5. Points within the surface represent the states of nonyielding.

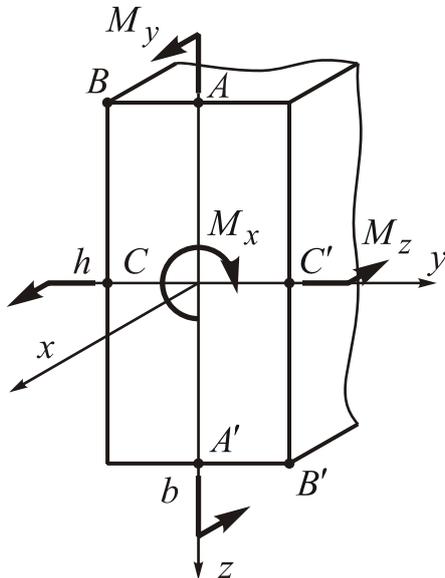
The von Mises theory agrees best with the test data for ductile materials and is in common use in design.

**Note. Of the failure criteria, the Tresca is the most conservative for all materials, the von Mises the most representative for ductile materials, and the Rankine the best fit for brittle materials.**



## EXAMPLES

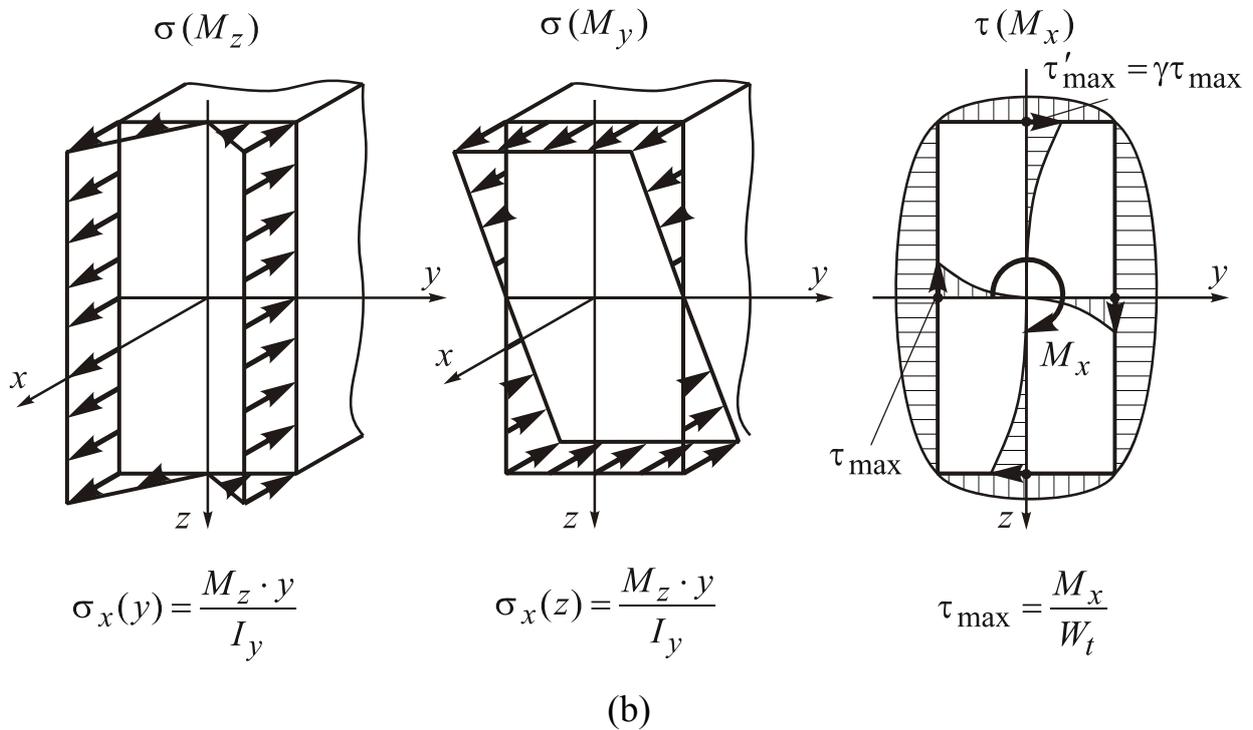
### Example 6.1



(a)

The rectangular beam of the cross-section shown in figure (a) is loaded by two bending moments  $M_y = 7.5 \text{ kN}\cdot\text{m}$ ,  $M_z = 3.2 \text{ kN}\cdot\text{m}$  acting respectively in vertical and horizontal planes of cross-sectional symmetry, and also by torque moment  $M_x = 7.0 \text{ kN}\cdot\text{m}$ . Assuming the allowable stress of beam material  $\sigma_{all} = 160 \text{ MPa}$ , find cross-sectional dimensions  $h$  and  $b$  taking into account that  $h/b = 2$  and maximum shear stress theory of failure is used.

**Solution** The combination of internal stresses produced by each separate internal force is shown in the figure (b). The table of the coefficients of sectional moduli  $W_i = \alpha hb^2$  allows to find the values of  $\alpha = 0.246$  and  $\gamma = 0.795$  in calculation of maximum shear stresses (see figure (b)).

**Table**

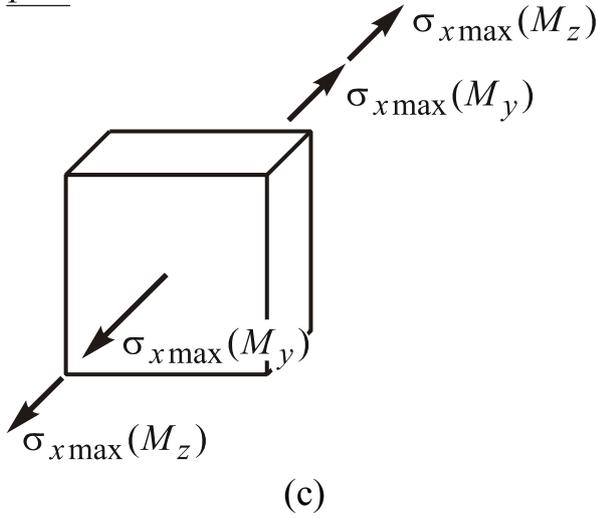
$\frac{h}{b}$	1	1.5	2.0	3.0	4.0	6.0	8.0	10.0	>10
$\alpha$	0.208	0.231	0.246	0.267	0.282	0.299	0.307	0.313	0.333
$\beta$	0.141	0.196	0.229	0.263	0.281	0.299	0.307	0.313	0.333
$\gamma$	1	0.859	0.795	0.753	0.745	0.743	0.743	0.743	0.743

The general problem is to find potentially dangerous points of the section from the viewpoint of combined action of the normal and shear stresses and write the conditions of strength in each of them taking into account the type of stress state at the point (uniaxial or combined).

As may be seen from figure (b) the maximum normal stress should act in two corner points  $B$  and  $B'$ . At these points

$$\sigma_{x_{\max}} = \sigma_B = \frac{M_y}{W_y} + \frac{M_z}{W_z}. \quad (\text{a})$$

p. B



**Note, that the shearing stresses at the points B and B' are equal to zero:**

$$\tau_{p.B} = 0. \quad (b)$$

Due to linear (uniaxial) stress-state at the points B and B' the condition of strength is really comparing of maximum acting normal stress with allowable one:

$$\sigma_{x_{max}} = \frac{M_y}{W_y} + \frac{M_z}{W_z} \leq \sigma_{all}, \quad (c)$$

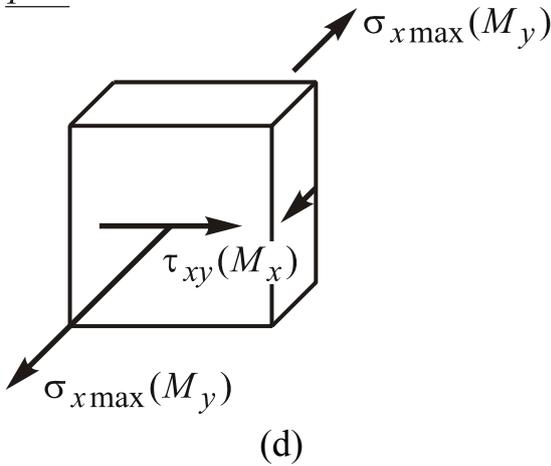
where  $W_y = bh^2/6$ ,  $W_z = hb^2/6$ . From

this condition the first of three possible values of the cross-sectional width  $b_1$  is

$$b_1 \geq \sqrt[3]{\frac{3M_y/2 + 3M_z}{\sigma_{all}}} = \sqrt[3]{\frac{3 \times 7.5 \times 10^3/2 + 3 \times 3.2 \times 10^3}{160 \times 10^6}} = 5.07 \times 10^{-2} \text{ m.}$$

The other two potentially dangerous points are the points A (or A') and C (or C'):

p. A



$$\sigma_x = \frac{M_y}{W_y}, \quad (d)$$

$$\tau_{xy} = \tau'_{max} = \gamma \frac{M_x}{W_t} = \gamma \frac{M_x}{\alpha hb^2}. \quad (e)$$

Determine the principal stresses:

$$\sigma_{1,2(3)} = \frac{\sigma_x}{2} \pm \frac{1}{2} \sqrt{\sigma_x^2 + 4\tau_{xy}^2}.$$

$$\sigma_1 = \frac{\sigma_x}{2} + \frac{1}{2} \sqrt{\sigma_x^2 + 4\tau_{xy}^2}, \quad \sigma_2 = 0,$$

$$\sigma_3 = \frac{\sigma_x}{2} - \frac{1}{2} \sqrt{\sigma_x^2 + 4\tau_{xy}^2}. \quad (f)$$

It should be observed that there is biaxial stressed state at the point A. According to the maximum shearing stress theory of failure (third theory of failure) the condition of strength is

$$\sigma_{eq}^{III} = (\sigma_1 - \sigma_3) \leq \sigma_{all}, \quad (g)$$

or

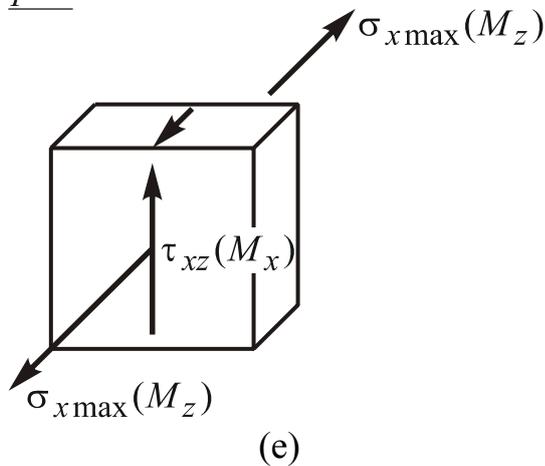
$$\sigma_{eq}^{III} = \sqrt{\sigma_x^2 + 4\tau_{xy}^2} \leq \sigma_{all}, \quad \text{or} \quad \sigma_{eq}^{III} = \sqrt{\left(\frac{M_y}{W_y}\right)^2 + 4\left(\gamma \frac{M_x}{\alpha hb^2}\right)^2} \leq \sigma_{all}. \quad (h)$$

From this condition second of three possible values of the cross-sectional width  $b_2$  will be

$$b_2 \geq \sqrt[3]{\frac{\sqrt{(3 \times M_y/2)^2 + (\gamma M_x/\alpha)^2}}{\sigma_{all}}} =$$

$$= \sqrt[3]{\frac{\sqrt{(3/2 \times 7.5 \times 10^3)^2 + (0.795 \times 7.0 \times 10^3/0.246)^2}}{160 \times 10^6}} = 5.40 \times 10^{-2} \text{ m.}$$

p. C



Point  $C$  is the last potentially dangerous point of the cross-section. Its stress state is biaxial because  $\sigma_x = M_z/W_z$  and  $\tau_{xz} = \tau_{\max} = M_x/\alpha hb^2$ . That is why we will use the maximum shear stress theory of failure to write the condition of strength by analogy with the point  $A$ :

$$\sigma_{eq}^{III} = \sqrt{\left(\frac{M_z}{W_z}\right)^2 + 4\left(\frac{M_x}{\alpha hb^2}\right)^2} \leq \sigma_{all}, \quad (i)$$

Taking into account that fact that  $W_z = hb^2/6$  we have

$$b_3 \geq \sqrt[3]{\frac{\sqrt{9(M_z)^2 + (M_x/\alpha)^2}}{\sigma_{all}}} =$$

$$= \sqrt[3]{\frac{\sqrt{9(3.2 \times 10^3)^2 + (7.0 \times 10^3/0.246)^2}}{160 \times 10^6}} = 5.725 \times 10^{-2} \text{ m.}$$

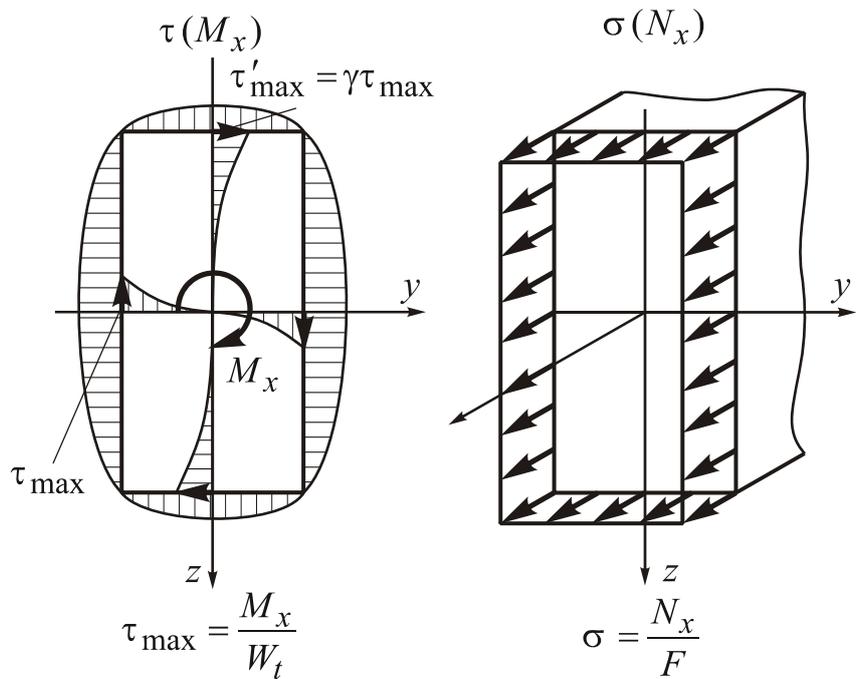
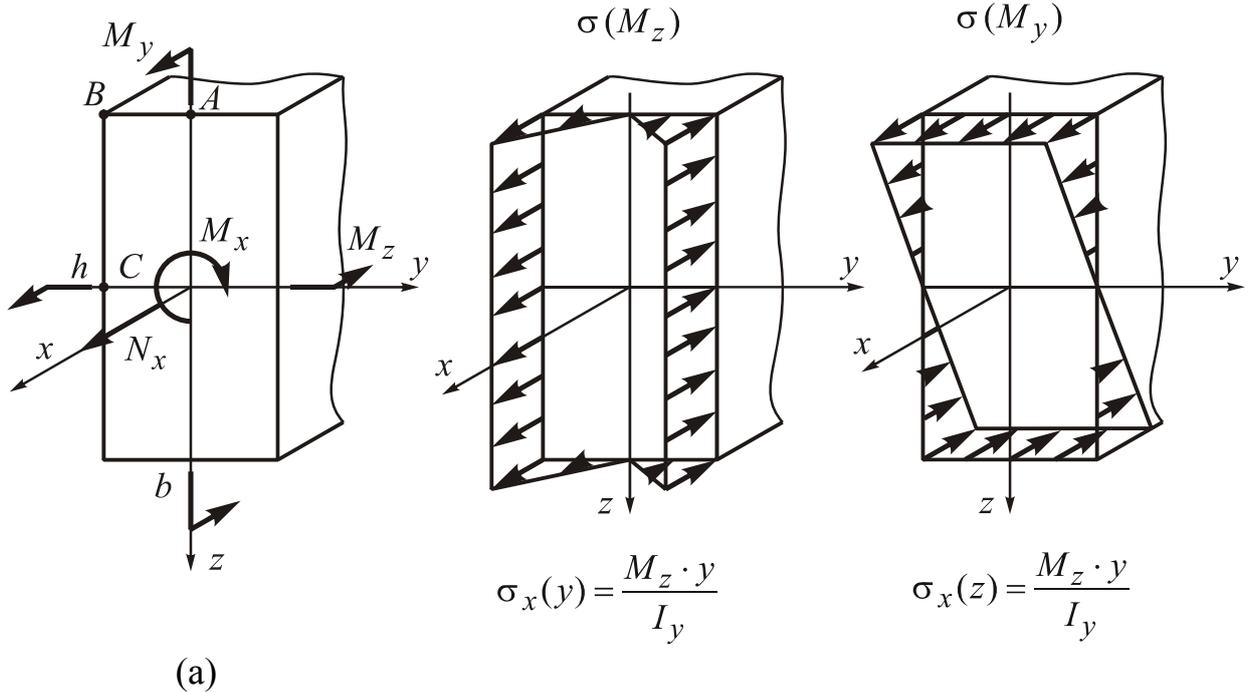
In result of comparing of three values of  $b$  the largest one is proposed as the cross-sectional width:  $b_{\max} = b_3 = 5.725 \times 10^{-2} \text{ m}$ ,  $h_{\max} = 11.46 \times 10^{-2} \text{ m}$ .

### Example 6.2

The rectangular beam of the cross-section shown in figure (a) is loaded by two bending moments  $M_y = 7.5 \text{ kN}\cdot\text{m}$ ,  $M_z = 3.2 \text{ kN}\cdot\text{m}$  acting respectively in vertical and horizontal planes of cross-sectional symmetry, and also by torque moment  $M_x = 7.0 \text{ kN}\cdot\text{m}$  and normal force  $N_x = 2.0 \text{ kN}$ . Assuming the allowable stress of beam material  $\sigma_{all} = 160 \text{ MPa}$ , find cross-sectional dimensions  $h$  and  $b$

taking into account that  $h/b=2$  and maximum shear stress theory of failure is used.

**Solution** The combination of internal stresses produced by each separate internal force is shown in figure (b). The table of the coefficients of sectional moduli  $W_t = \alpha hb^2$  allows to find the values of  $\alpha=0.246$  and  $\gamma=0.795$  in calculation of maximum shear stresses (see figure (b)).



(b)

**Table**

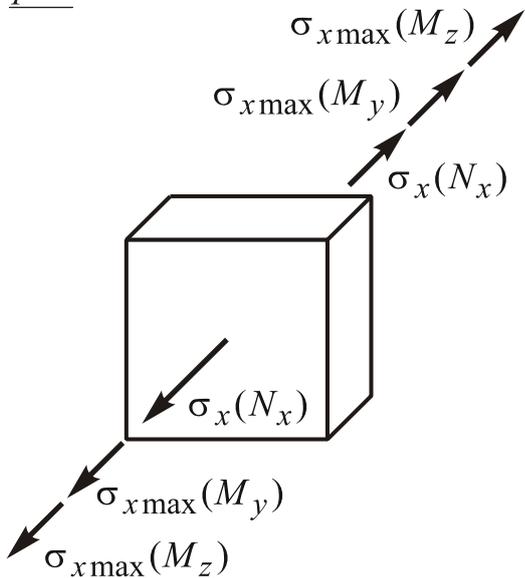
$\frac{h}{b}$	1	1.5	2.0	3.0	4.0	6.0	8.0	10.0	>10
$\alpha$	0.208	0.231	0.246	0.267	0.282	0.299	0.307	0.313	0.333
$\beta$	0.141	0.196	0.229	0.263	0.281	0.299	0.307	0.313	0.333
$\gamma$	1	0.859	0.795	0.753	0.745	0.743	0.743	0.743	0.743

First of all, it is necessary to find potentially dangerous points of the section from the viewpoint of combined action of the normal and shear stresses and write the conditions of strength in these points taking into account the type of stress state at the point (uniaxial or combined).

As may be seen from figure (b) the maximum normal stress should act in unique corner point  $B$ . At this point

$$\sigma_{x_{\max}} = \sigma_B = \frac{N_x}{A} + \frac{M_y}{W_y} + \frac{M_z}{W_z}. \tag{a}$$

$p. B$



(c)

**Note, that the shearing stress at the point  $B$  is equal to zero:**

$$\tau_{p.B} = 0. \tag{b}$$

Due to linear (uniaxial) stress-state at the points  $B$  and  $B'$  the condition of strength is the really comparing of maximum acting normal stress with allowable one:

$$\sigma_{x_{\max}} = \frac{N_x}{A} + \frac{M_y}{W_y} + \frac{M_z}{W_z} \leq \sigma_{all}, \tag{c}$$

where  $W_y = bh^2/6$ ,  $W_z = hb^2/6$ .

To simplify the solution we will solve the problem in two approaches. Rough result may be obtained neglecting  $\sigma(N_x)$ . Then the first of three possible values of the cross-sectional width  $b_1$  is

$$b_1 > \sqrt[3]{\frac{3M_y/2 + 3M_z}{\sigma_{all}}} = \sqrt[3]{\frac{3 \times 7.5 \times 10^3 / 2 + 3 \times 3.2 \times 10^3}{160 \times 10^6}} = 5.07 \times 10^{-2} \text{ m.}$$

In second approach check the strength of point  $B$  comparing actual stress in accordance with formula (c) with allowable one:

$$\begin{aligned}\sigma_{x_{\max}} = \sigma_B &= \frac{N_x}{A} + \frac{M_y}{W_y} + \frac{M_z}{W_z} = \frac{N_x}{bh} + \frac{6M_y}{bh^2} + \frac{6M_z}{hb^2} = \frac{N_x}{2b^2} + \frac{3M_y}{2b^3} + \frac{3M_z}{b^3} = \\ &= \frac{2.0 \times 10^3}{2(5.07 \times 10^{-2})^2} + \frac{3 \times 7.5 \times 10^3}{2(5.07 \times 10^{-2})^3} + \frac{3 \times 3.2 \times 10^3}{(5.07 \times 10^{-2})^3} = \\ &= (3.89 + 86.33 + 73.67) \times 10^6 = 163.89 \times 10^6 \text{ Pa} = 163.89 \text{ MPa}.\end{aligned}$$

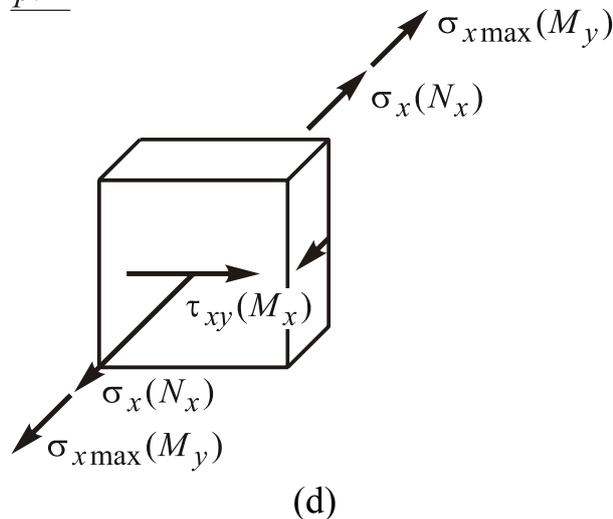
Let us estimate the overstress of the point  $B$ :

$$\Delta\sigma = \frac{\sigma_{x_{\max}} - \sigma_{all}}{\sigma_{all}} = \frac{163.89 - 160}{160} = 0.0243 = 2.43\%.$$

In applied mechanics, 5% overstress is permissible. It means that *the width  $b_1 = 5.07 \text{ cm}$  is final from the viewpoint of the strength of the point  $B$ .*

The other two potentially dangerous points are  $A$  and  $C$  points. In the first approach, we will neglect  $\sigma(N_x)$  in both points and, first of all, find rough values of the width  $b$ . In second approach, we will estimate the overstress of these points produced by  $\sigma(N_x)$ , because 5% overstress is permissible in applied mechanics.

*p. A*



$$\sigma_x = \frac{N_x}{A} + \frac{M_y}{W_y}; \quad (d)$$

$$\tau_{xy} = \tau'_{\max} = \gamma \frac{M_x}{\alpha hb^2}. \quad (e)$$

Because of biaxial stressed state at the point  $A$ , according to the maximum shearing stress theory of failure condition of strength is

$$\sigma_{eq}^{III} = (\sigma_1 - \sigma_3) \leq \sigma_{all}, \quad (f)$$

or, in the first approach,

$$\sigma_{eq}^{III} = \sqrt{\sigma_x^2 + 4\tau_{xy}^2} \leq \sigma_{all}, \quad \text{or} \quad \sigma_{eq}^{III} = \sqrt{\left(\frac{M_y}{W_y}\right)^2 + 4\left(\gamma \frac{M_x}{\alpha hb^2}\right)^2} \leq \sigma_{all}. \quad (g)$$

In result, rough value of cross-sectional width is equal to

$$b_2 \geq \sqrt[3]{\frac{\sqrt{\left(3 \times M_y / 2\right)^2 + \left(\gamma M_x / \alpha\right)^2}}{\sigma_{all}}} =$$

$$= \sqrt[3]{\frac{\sqrt{\left(3/2 \times 7.5 \times 10^3\right)^2 + \left(0.795 \times 7.0 \times 10^3 / 0.246\right)^2}}{160 \times 10^6}} = 5.40 \times 10^{-2} \text{ m.}$$

In second approach, calculate the overstress of the point  $A$ , produced by  $\sigma(N_x)$  and estimate it. Actual value of the equivalent stress is

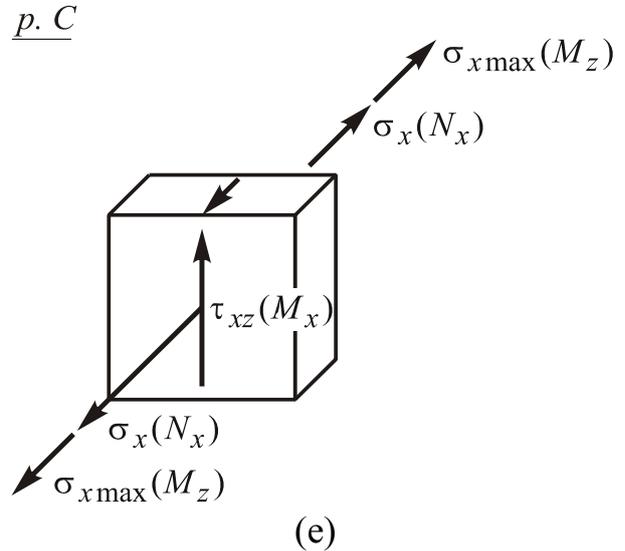
$$\begin{aligned} \sigma_{eq}^{III} &= \sqrt{\left(\frac{N_x}{A} + \frac{M_y}{W_y}\right)^2 + 4\left(\gamma \frac{M_x}{\alpha hb^2}\right)^2} = \sqrt{\left(\frac{N_x}{2b^2} + \frac{3M_y}{2b^3}\right)^2 + \left(\gamma \frac{M_x}{\alpha b^3}\right)^2} = \\ &= \sqrt{\left(\frac{2.0 \times 10^3}{2(5.40 \times 10^{-2})^2} + \frac{3 \times 7.5 \times 10^3}{2(5.40 \times 10^{-2})^3}\right)^2 + \left(\frac{0.795 \times 7.0 \times 10^3}{0.246(5.40 \times 10^{-2})^3}\right)^2} = \\ &\sqrt{\left(0.34 \times 10^6 + 71.44 \times 10^6\right)^2 + \left(143.66 \times 10^6\right)^2} = 160.59 \text{ MPa.} \end{aligned}$$

$$\text{Overstress } \Delta\sigma = \frac{\sigma_{eq}^{III} - \sigma_{all}}{\sigma_{all}} = \frac{160.59 - 160}{160} = 0.0037 = 0.37\%.$$

The overstress  $\Delta\sigma < 5\%$ . It is permissible and the value  $b_2 = 5.40$  cm is final from the viewpoint of the strength of point  $A$ .

Point  $C$  is the last potentially dangerous point of the cross-section. Its stress state is biaxial because  $\sigma_x = N_x/A + M_z/W_z$ ,  $\tau_{xz} = \tau_{max} = M_x/(\alpha hb^2)$ . That is why we will use the maximum shear stress theory of failure to write the condition of strength by analogy with the point  $A$ :

$$\sigma_{eq}^{III} = \sqrt{\left(\frac{N_x}{A} + \frac{M_z}{W_z}\right)^2 + 4\left(\frac{M_x}{\alpha hb^2}\right)^2} \leq \sigma_{all}.$$



Taking into account that fact that  $W_z = hb^2/6$  in the first approach calculate the rough value of cross-sectional width ignoring  $\sigma(N_x)$ , i.e. using the formula

$$b_3 \geq \sqrt[3]{\frac{\sqrt{9(M_z)^2 + (M_x/\alpha)^2}}{\sigma_{all}}} =$$

$$= \sqrt[3]{\frac{\sqrt{9(3.2 \times 10^3)^2 + (7.0 \times 10^3 / 0.246)^2}}{160 \times 10^6}} = 5.725 \times 10^{-2} \text{ m.}$$

In second approach, calculate the overstress of the point  $C$ , produced by  $\sigma(N_x)$  and estimate it. For this, calculate actual value of equivalent stress in the point substituting  $b_3$  value into (h) formula:

$$\begin{aligned} \sigma_{eq}^{III} &= \sqrt{\left(\frac{N_x}{2b^2} + \frac{3M_z}{b^3}\right)^2 + \left(\frac{M_x}{\alpha b^3}\right)^2} = \\ &= \sqrt{\left(\frac{2.0 \times 10^3}{2(5.725 \times 10^{-2})^2} + \frac{3 \times 3.2 \times 10^3}{(5.725 \times 10^{-2})^3}\right)^2 + \left(\frac{7.0 \times 10^3}{0.246(5.725 \times 10^{-2})^3}\right)^2} = \\ &= \sqrt{(0.305 \times 10^6 + 51.16 \times 10^6)^2 + (151.65 \times 10^6)^2} = 160.14 \times 10^6 \text{ Pa} = 160.14 \text{ MPa.} \\ \text{Overstress } \Delta\sigma &= \frac{\sigma_{eq}^{III} - \sigma_{all}}{\sigma_{all}} = \frac{160.14 - 160}{160} = 0.00088 = 0.088\%. \end{aligned}$$

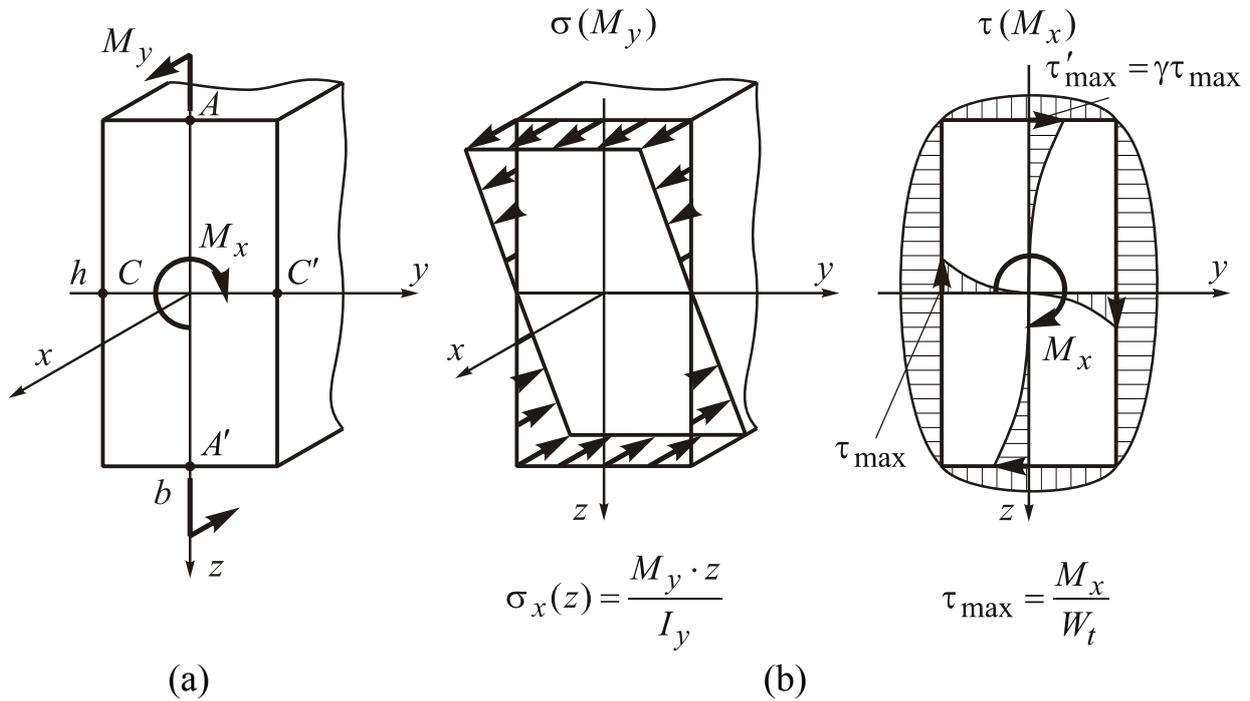
The overstress  $\Delta\sigma < 5\%$ . It is permissible and the value  $b_3 = 5.725 \text{ cm}$  is final from the viewpoint of the strength of point  $C$ .

In result of comparing of three actual values of  $b$  the largest one is proposed as the cross-sectional width:  $b_{\max} = b_3 = 5.725 \times 10^{-2} \text{ m}$ ,  $h_{\max} = 11.45 \times 10^{-2} \text{ m}$ .

### Example 6.3

The rectangular beam of the cross-section shown in figure (a) is loaded by bending moment  $M_y = 7.5 \text{ kN}\cdot\text{m}$ , and also torque moment  $M_x = 7.0 \text{ kN}\cdot\text{m}$ . Assuming the allowable stress of beam material  $\sigma_{all} = 160 \text{ MPa}$ , find cross-sectional dimensions  $h$  and  $b$  taking into account that  $h/b = 2$  and maximum shear stress theory of failure is used.

**Solution** The combination of internal stresses produced by each separate internal force is shown in figure (b). The table of the coefficients of sectional moduli  $W_t = \alpha hb^2$  allows to find the values of  $\alpha = 0.246$  and  $\gamma = 0.795$  in calculation of maximum shear stresses (see figure (b)).



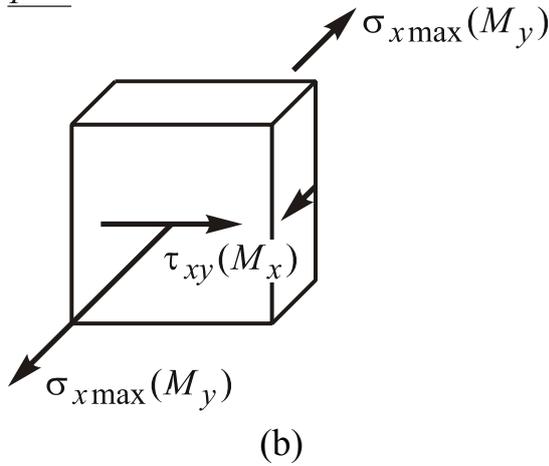
**Table**

$\frac{h}{b}$	1	1.5	2.0	3.0	4.0	6.0	8.0	10.0	>10
$\alpha$	0.208	0.231	0.246	0.267	0.282	0.299	0.307	0.313	0.333
$\beta$	0.141	0.196	0.229	0.263	0.281	0.299	0.307	0.313	0.333
$\gamma$	1	0.859	0.795	0.753	0.745	0.743	0.743	0.743	0.743

Potentially dangerous points of the section from the viewpoint of combined action of the normal and shear stresses may be found in result of analysis of internal stresses produced by separate internal force shown in the figure (a). Note that *A* and *A'* points are two equidangerous points of maximum normal stresses and also *C* and *C'* points are two equidangerous points of maximum shear stresses. Totally, the points *A* and *C* (or *A'* and *C'*) are two potentially dangerous points under this specified loading. The conditions of strength will be written taking into account the combined stress state in these points.

As may be seen from figure (a) maximum normal stress should act in the point *A* simultaneously with relatively large shear stress (see figure (b)).

p. A



$$\sigma_{x\max} = \frac{M_y}{W_y}; \tag{a}$$

$$\tau_{xy} = \tau'_{\max} \frac{M_x}{W_t} = \gamma \frac{M_x}{\alpha hb^2}. \tag{b}$$

Due to biaxial stressed state at the point *A* and in accordance with the maximum shearing stress theory of failure condition of strength may be written as

$$\sigma_{eq}^{III} = (\sigma_1 - \sigma_3) \leq \sigma_{all} \tag{g}$$

or

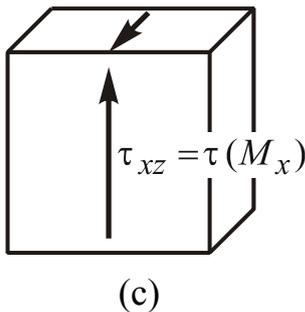
$$\sigma_{eq}^{III} = \sqrt{\sigma_x^2 + 4\tau_{xy}^2} \leq \sigma_{all}, \quad \text{or} \quad \sigma_{eq}^{III} = \sqrt{\left(\frac{M_y}{W_y}\right)^2 + 4\left(\gamma \frac{M_x}{\alpha hb^2}\right)^2} \leq \sigma_{all}. \tag{h}$$

$$b_1 \geq \sqrt[3]{\frac{\sqrt{(3 \times M_y / 2)^2 + (\gamma M_x / \alpha)^2}}{\sigma_{all}}} = \sqrt[3]{\frac{\sqrt{(3/2 \times 7.5 \times 10^3)^2 + (0.795 \times 7.0 \times 10^3 / 0.246)^2}}{160 \times 10^6}} = 5.40 \times 10^{-2} \text{ m.}$$

Point *C* is second potentially dangerous point of the cross-section. Its stress state is pure shear, which is really an important particularity of biaxial stress state. Acting stress  $\tau_{xz} = \tau_{\max} = M_x / (\alpha hb^2)$ . That is why we will use the maximum shear stress theory of failure to write the condition of strength by analogy with the point *A*:

p. C

$$\sigma_{eq}^{III} = \sqrt{0 + 4\left(\frac{M_x}{\alpha hb^2}\right)^2} \leq \sigma_{all}. \tag{i}$$



From this

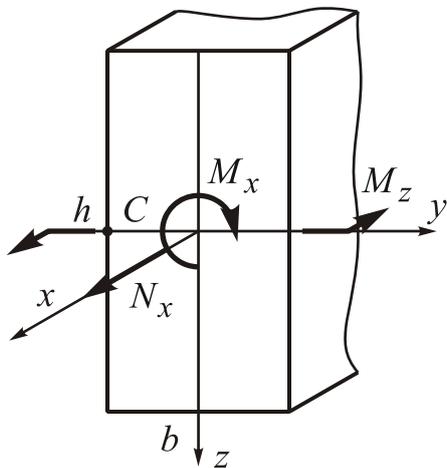
$$b_2 \geq \sqrt[3]{\frac{M_x / \alpha}{\sigma_{all}}} = \sqrt[3]{\frac{7.0 \times 10^3 / 0.246}{160 \times 10^6}} = 5.62 \times 10^{-2} \text{ m.}$$

In result of comparing of two values of *b* the largest one is proposed as the cross-sectional width:

$$b_{\max} = b_2 = 5.62 \times 10^{-2} \text{ m, } h_{\max} = 11.24 \times 10^{-2} \text{ m.}$$

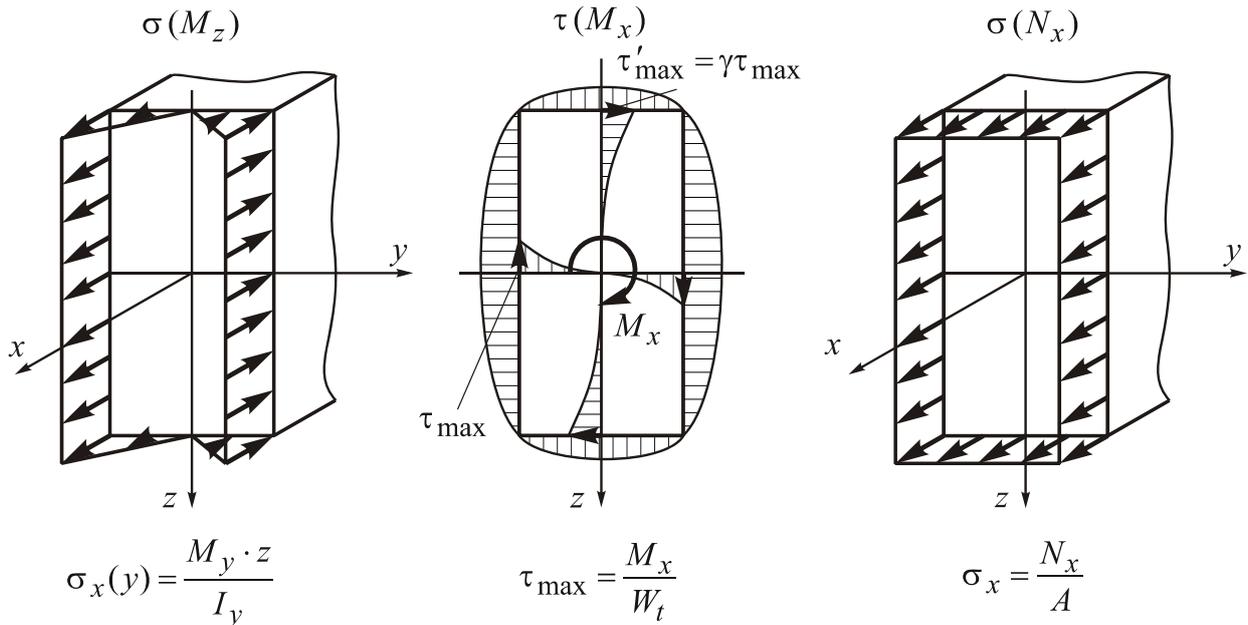
**Example 6.4**

The rectangular bar of the cross-section shown in figure (a) is loaded by bending moment  $M_z = 3.2 \text{ kN}\cdot\text{m}$  acting in horizontal plane of cross-sectional symmetry, and also by torque moment  $M_x = 7.0 \text{ kN}\cdot\text{m}$  and normal force  $N_x = 2.0 \text{ kN}$ . Assuming the allowable stress of beam material  $\sigma_{all} = 160 \text{ MPa}$ , find cross-sectional dimensions  $h$  and  $b$  taking into account that  $h/b = 2$  and maximum shear stress theory of failure is used.



(a)  
 $\sigma(M_z)$

**Solution** The combination of internal stresses produced by each separate internal force is shown in figure (b). The table of the coefficients of sectional moduli  $W_t = \alpha hb^2$  allows to find the values of  $\alpha = 0.246$  and  $\gamma = 0.795$  in calculation of maximum shear stresses (see figure (b)).



(b)

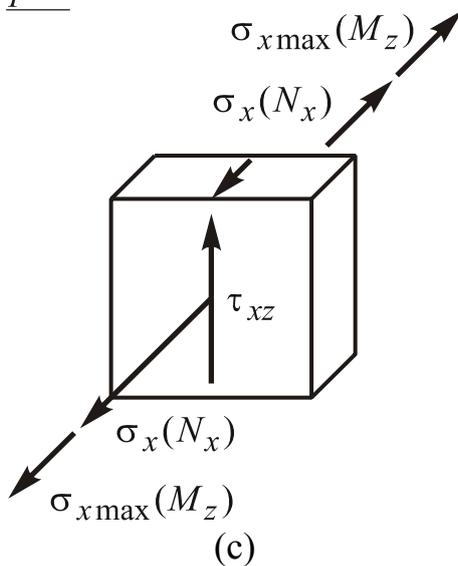
**Table**

$\frac{h}{b}$	1	1.5	2.0	3.0	4.0	6.0	8.0	10.0	>10
$\alpha$	0.208	0.231	0.246	0.267	0.282	0.299	0.307	0.313	0.333
$\beta$	0.141	0.196	0.229	0.263	0.281	0.299	0.307	0.313	0.333
$\gamma$	1	0.859	0.795	0.753	0.745	0.743	0.743	0.743	0.743

First of all, it is necessary to find potentially dangerous points of the section from the viewpoint of combined action of the normal and shear stresses and write the conditions of strength in these points taking into account the type of stress state at the point (uniaxial or combined). It is evident that unique dangerous point  $C$  exists in this type of loading, because it is simultaneously the point of maximum normal and shear stresses (see figure (b)).

To simplify the solution we will solve the problem in two approaches. In the first approach, we will neglect  $\sigma(N_x)$  in the point and find rough value of the width  $b$ . In second approach, we will estimate the overstress of this point produced by  $\sigma(N_x)$ , because 5% overstress is permissible in applied mechanics.

p. C



The stress state in point  $C$  is biaxial because

$$\sigma_x = N_x/A + M_z/W_z,$$

$\tau_{xz} = \tau_{\max} = M_x/(\alpha hb^2)$ . That is why we will use the maximum shear stress theory of failure to write the condition of strength:

$$\sigma_{eq}^{III} = \sqrt{\left(\frac{N_x}{A} + \frac{M_z}{W_z}\right)^2 + 4\left(\frac{M_x}{\alpha hb^2}\right)^2} \leq \sigma_{all}. \quad (a)$$

Taking into account that fact that  $W_z = hb^2/6$  in the first approach calculate the rough value of cross-sectional width ignoring  $\sigma(N_x)$ , i.e. using the formula

$$b \geq \sqrt[3]{\frac{\sqrt{9(M_z)^2 + (M_x/\alpha)^2}}{\sigma_{all}}} = \quad (b)$$

$$= \sqrt[3]{\frac{\sqrt{9(3.2 \times 10^3)^2 + (7.0 \times 10^3/0.246)^2}}{160 \times 10^6}} = 5.725 \times 10^{-2} \text{ m.}$$

In second approach, calculate the overstress of the point  $C$ , produced by  $\sigma(N_x)$  and estimate it. For this, calculate actual value of equivalent stress in the point substituting  $b$  value into (a) formula:

$$\begin{aligned} \sigma_{eq}^{III} &= \sqrt{\left(\frac{N_x}{2b^2} + \frac{3M_z}{b^3}\right)^2 + \left(\frac{M_x}{\alpha b^3}\right)^2} = \quad (c) \\ &= \sqrt{\left(\frac{2.0 \times 10^3}{2(5.725 \times 10^{-2})^2} + \frac{3 \times 3.2 \times 10^3}{(5.725 \times 10^{-2})^3}\right)^2 + \left(\frac{7.0 \times 10^3}{0.246(5.725 \times 10^{-2})^3}\right)^2} = \end{aligned}$$

$$= \sqrt{(0.305 \times 10^6 + 51.16 \times 10^6)^2 + (151.65 \times 10^6)^2} = 160.14 \times 10^6 \text{ Pa} = 160.14 \text{ MPa}.$$

$$\text{Overstress } \Delta\sigma = \frac{\sigma_{eq}^{\text{III}} - \sigma_{all}}{\sigma_{all}} = \frac{160.14 - 160}{160} = 0.00088 = 0.088\%.$$

The overstress  $\Delta\sigma < 5\%$ . It is permissible and *the value  $b = 5.725 \text{ cm}$  is final from the viewpoint of the strength of point C*. It is proposed as the cross-sectional width:  $b = 5.725 \times 10^{-2} \text{ m}$ ,  $h = 11.45 \times 10^{-2} \text{ m}$ .

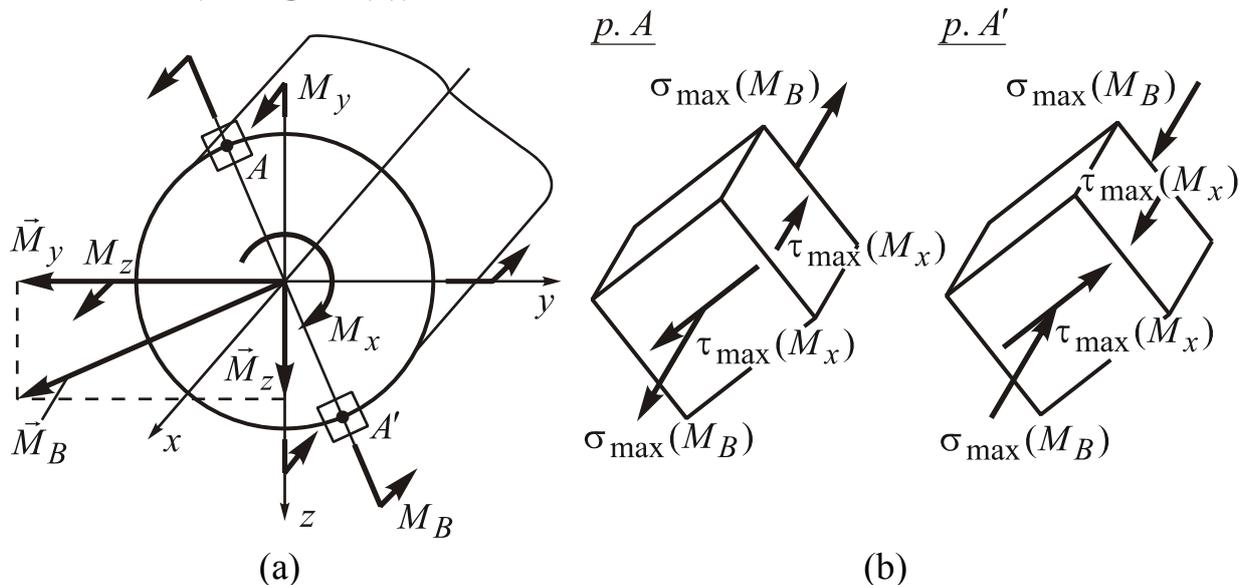
### Example 6.5

The round solid rod of cross-section shown in the figure (a) is loaded by two bending moments  $M_y = 7.5 \text{ kN}\cdot\text{m}$ ,  $M_z = 3.2 \text{ kN}\cdot\text{m}$  acting respectively in vertical and horizontal planes of cross-sectional symmetry, and also by torque moment  $M_x = 7.0 \text{ kN}\cdot\text{m}$ . Assuming that allowable stress of beam material  $\sigma_{all} = 160 \text{ MPa}$ , find cross-sectional diameter  $d$  taking into account that maximum shear stress theory is used.

**Solution** To find dangerous points of cross-section taking into account its polar symmetry first of all calculate resultant bending moment  $M_B$ . Its scalar value is

$$M_B = \sqrt{M_y^2 + M_z^2} = \sqrt{(7.5 \times 10^3)^2 + (3.2 \times 10^3)^2} = 8.154 \times 10^3 \text{ kN}\cdot\text{m}. \quad (\text{a})$$

The points of maximum normal stresses (equidangerous points  $A$  and  $A'$  in the figure (a)) are really the points of  $M_B$  moment plane and cross-section contour intersection (see figure (b)).



In these equidangerous points  $\sigma_{x \max} = M_B/W_{n.a.}$  and  $\tau_{\max} = M_x/W_\rho$ . It means that the plane stress state is realized and the maximum shearing stress theory of failure is necessary to use to create condition of strength:

$$\sigma_{eq}^{III} = \sqrt{\left(\frac{M_B}{W_{n.a.}}\right)^2 + 4\left(\frac{M_x}{W_\rho}\right)^2} \leq \sigma_{all}.$$

Because  $W_\rho = 2W_{n.a.}$ ,

$$\sigma_{eq}^{III} = \frac{\sqrt{M_x^2 + M_y^2 + M_z^2}}{W_{n.a.}},$$

or

$$\sigma_{eq}^{III} = \frac{M^{III}}{W_{n.a.}} \leq \sigma_{all}, \quad (7)$$

where

$$\begin{aligned} M^{III} &= \sqrt{M_x^2 + M_y^2 + M_z^2} = \\ &= \sqrt{(7.0 \times 10^3)^2 + (7.5 \times 10^3)^2 + (3.2 \times 10^3)^2} = 10.747 \times 10^3 \text{ kN}\cdot\text{m}. \end{aligned}$$

Taking into account that  $W_{n.a.} = \pi d^3/32$  the resultant formula for  $d$  is

$$d \geq \sqrt[3]{\frac{32M^{III}}{\pi\sigma_{all}}} = \sqrt[3]{\frac{32 \times 10.747 \times 10^3}{3.14 \times 160 \times 10^6}} = 8.813 \times 10^{-2} \text{ m}.$$

### Example 6.6

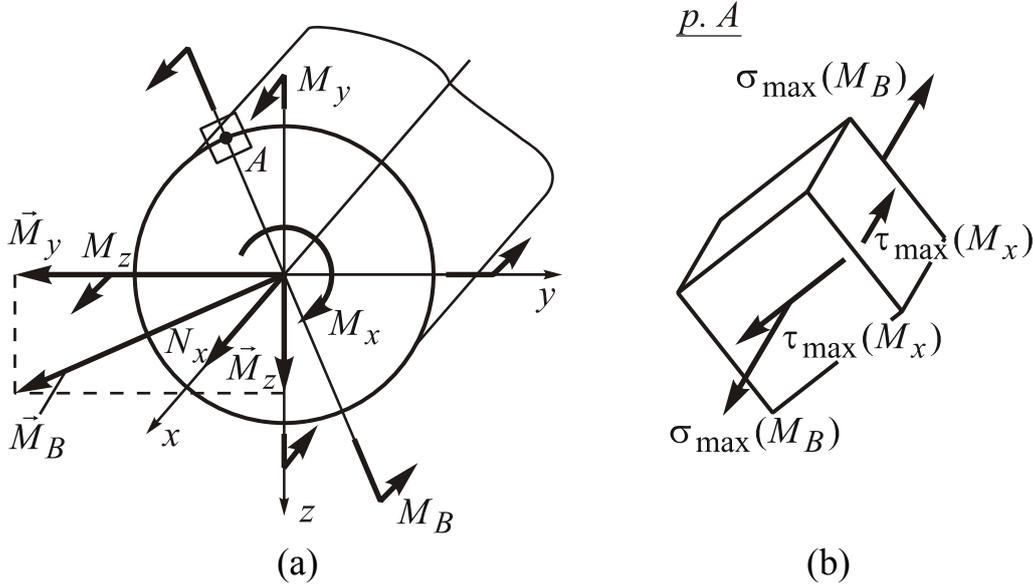
The round solid rod of cross-section shown in the figure (a) is loaded by two bending moments  $M_y = 7.5 \text{ kN}\cdot\text{m}$ ,  $M_z = 3.2 \text{ kN}\cdot\text{m}$  acting respectively in vertical and horizontal planes of cross-sectional symmetry, and also by torque moment  $M_x = 7.0 \text{ kN}\cdot\text{m}$  and normal force  $N_x = 2.0 \text{ kN}$ . Assuming that allowable stress of beam material  $\sigma_{all} = 160 \text{ MPa}$ , find cross-sectional diameter  $d$  taking into account that maximum shear stress theory is used.

**Solution** To find critical points of cross-section taking into account its polar symmetry first of all calculate resultant bending moment  $M_B$ . Its scalar value is

$$M_B = \sqrt{M_y^2 + M_z^2} = \sqrt{(7.5 \times 10^3)^2 + (3.2 \times 10^3)^2} = 8.154 \times 10^3 \text{ kN}\cdot\text{m}. \quad (a)$$

Because of the  $N_x$  normal force, unique dangerous point  $A$  exists. It is one or two points of maximum bending normal stresses (points  $A$  and  $A'$  in the figure (a)),

which are really the points of  $M_B$  moment plane and cross-section contour intersection (see figure (b)).



In the point  $A$  plane stress state is realized because  $\sigma_A = \sigma_{\max} = N_A / A + M_B / W_{n.a.}$  and  $\tau_A = \tau_{\max} = M_x / W_\rho$ . That is why the maximum shearing stress theory of failure is necessary to use to create condition of strength:

$$\sigma_{eq}^{III} = \sqrt{\left(\frac{N_x}{A} + \frac{M_B}{W_{n.a.}}\right)^2 + 4\left(\frac{M_x}{W_\rho}\right)^2} \leq \sigma_{all}, \quad W_\rho = 2W_{n.a.} \quad (a)$$

To simplify the solution we will solve the problem in two approaches. In the first approach, we will neglect  $\sigma(N_x)$  in the point and find rough value of the diameter  $d$ . In second approach, we will estimate the overstress of this point produced by  $\sigma(N_x)$ , because 5% overstress is permissible in applied mechanics.

It means that simplified condition of strength will be

$$\sigma_{eq}^{III} = \sqrt{\left(\frac{M_B}{W_{n.a.}}\right)^2 + 4\left(\frac{M_x}{W_\rho}\right)^2} \leq \sigma_{all},$$

or

$$\sigma_{eq}^{III} = \frac{\sqrt{M_x^2 + M_y^2 + M_z^2}}{W_{n.a.}} \leq \sigma_{all},$$

or

$$\sigma_{eq_{p.A(p.B)}}^{III} = \frac{M^{III}}{W_{n.a.}} \leq \sigma_{all}$$

where

$$M^{III} = \sqrt{M_x^2 + M_y^2 + M_z^2} =$$

$$= \sqrt{(7.0 \times 10^3)^2 + (7.5 \times 10^3)^2 + (3.2 \times 10^3)^2} = 10.747 \times 10^3 \text{ kN}\cdot\text{m}.$$

Taking into account that  $W_{n.a.} = \pi d^3/32$  the resultant formula for  $d$  is

$$d \geq \sqrt[3]{\frac{32M^{III}}{\pi\sigma_{all}}} = \sqrt[3]{\frac{32 \times 10.747 \times 10^3}{3.14 \times 160 \times 10^6}} = 8.813 \times 10^{-2} \text{ m}.$$

In second approach, calculate the overstress of the point  $A$ , produced by  $\sigma(N_x)$  and estimate it. For this, calculate actual value of equivalent stress in the point substituting  $d$  value into (a) formula:

$$\begin{aligned} \sigma_{eq}^{III} &= \sqrt{\left(\frac{N_x}{A} + \frac{M_B}{W_{n.a.}}\right)^2 + 4\left(\frac{M_x}{W_\rho}\right)^2} = \\ &= \sqrt{\left(\frac{4N_x}{\pi d^2} + \frac{32\sqrt{M_y^2 + M_z^2}}{\pi d^3}\right)^2 + 4\left(\frac{16M_x}{\pi d^3}\right)^2} = \\ &= \sqrt{\left(\frac{(4 \times 2.0 \times 10^3)}{3.14 \times (8.813 \times 10^{-2})^2} + \frac{32\sqrt{(7.5 \times 10^3)^2 + (3.2 \times 10^3)^2}}{3.14 \times (8.813 \times 10^{-2})^3}\right)^2 +} \\ &\quad + 4\left(\frac{16 \times 7.0 \times 10^3}{3.14 \times (8.813 \times 10^{-2})^3}\right)^2} = 160.25 \text{ MPa} \end{aligned} \tag{b}$$

$$\text{Overstress } \Delta\sigma = \frac{\sigma_{eq}^{III} - \sigma_{all}}{\sigma_{all}} = \frac{160.25 - 160}{160} = 0.00156 = 0.156\%.$$

The overstress  $\Delta\sigma < 5\%$ . It is permissible and the value  $d = 8.813$  cm is final from the viewpoint of the strength of point  $A$ . It is proposed as the cross-sectional diameter.

### Example 6.7

A closed-end cylinder, 0.61 m in diameter and 12.7 mm thick, is fabricated of steel of tensile strength  $\sigma_y = 240$  MPa. Calculate the allowable pressure the shell can carry based upon a factor of safety  $f_s = 2$ . Apply the Tresca and von Mises yielding theories of failure.

**Solution** The circumferential, axial, and radial stresses are given by

$$\sigma_1 = \frac{pr}{t} = 24p, \quad \sigma_2 = \frac{pr}{2t} = 12p, \quad \sigma_3 = 0.$$

Insertion of these expressions into Eq. (6.5) will provide the critical pressures for the *maximum shearing stress theory*:

$$24p_{all} - 0 = \frac{1}{2}(240 \times 10^6),$$

$$p_{all} = 5.0 \text{ MPa.}$$

Insertion of these expressions into Eq. (6.10) will provide the critical pressures for the *maximum energy of distortion theory*:

$$p_{all} \left( 24^2 - 24 \times 12 + 12^2 \right)^{1/2} = \frac{1}{2} (240 \times 10^6),$$

$$p_{all} = 5.77 \text{ MPa.}$$

The permissible value of the internal pressure is thus limited to 5.0 MPa.

### Example 6.8

A circular shaft of diameter  $d$  and tensile yield strength  $\sigma_y$  is subjected to combined axial tensile force  $P$  and torque moment  $M_T$ . Determine the bending moment  $M_B$  that can also be applied simultaneously to the shaft. Use a factor of safety  $f_s$  and employ the maximum energy of distortion theory of failure. Compute the value of  $M_B$  for the following data:  $P = 100 \text{ kN}$ ,  $M_T = 5 \text{ kN}\cdot\text{m}$ ,  $d = 60 \text{ mm}$ ,  $\sigma_y = 300 \text{ MPa}$ , and  $f_s = 1.2$ .

**Solution** In this example, the outer-fiber stresses in the shaft are

$$\sigma = \frac{M_B}{W_{n.a.}} + \frac{P}{A} = \frac{32M_B}{\pi d^3} + \frac{4P}{\pi d^2}, \quad \tau = \frac{M_T}{W_\rho} = \frac{16M_T}{\pi d^3}. \quad (\text{a})$$

Basing on the equation (6.10) we will take into account that the principal stresses for members subjected to combined normal stress in tension and shear stress in torsion are

$$\sigma_{1,2} = \frac{\sigma}{2} \pm \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}. \quad (\text{b})$$

Using the maximum energy of distortion theory, and  $\sigma_{all} = \sigma_y / f_s$ , we obtain

$$\sigma^2 + 3\tau^2 = (\sigma_{all})^2. \quad (\text{c})$$

Introducing Eq. (a) into Eq. (c), we derive the governing expression:

$$\left(\frac{32M_B}{\pi d^3} + \frac{4P}{\pi d^2}\right)^2 + 3\left(\frac{16M_T}{\pi d^3}\right)^2 = (\sigma_{all})^2. \quad (d)$$

Upon simplification and rearrangement, the foregoing results in the quadratic equation in  $M_B$ :

$$M_B^2 + (0.25Pd)M_B + \left[0.75M_T^2 + 0.0156P^2d^2 - 0.0096(\sigma_{all})^2d^6\right] = 0.$$

The valid solution is therefore

$$M_B = -\frac{1}{2}c_1 + \frac{1}{2}\left[c_1^2 - 4(c_2 + c_3 - c_4)\right]^{1/2}, \quad (e)$$

where  $c_1 = 0.25Pd$ ,  $c_2 = 0.75M_T^2$ ,  $c_3 = 0.0156P^2d^2$ , and  $c_4 = 0.0096(\sigma_{all})^2d^6$ . When we substitute the numerical values, Eq. (e) leads to  $M_B = 2.29 \text{ kN}\cdot\text{m}$ .

### Example 6.9

A solid shaft is fitted with 300-mm-diameter pulleys, supported by frictionless bearings at  $A$  and  $B$ , and loaded as shown in figure (a). If  $\tau_{all} = 70 \text{ MPa}$ , calculate the required diameter of the shaft according to the maximum shear stress theory.

**Solution** A complete free-body diagram of the shaft is shown in figure (b). The determination of the largest value of  $(M_z^2 + M_y^2 + M_x^2)^{1/2}$  is facilitated by use of moments diagrams (figure (c)). At point  $C$ , we find

$$(M_z^2 + M_y^2 + M_x^2)^{1/2} = (1^2 + 5.5^2 + 0.6^2)^{1/2} = 5.62 \text{ kN}\cdot\text{m},$$

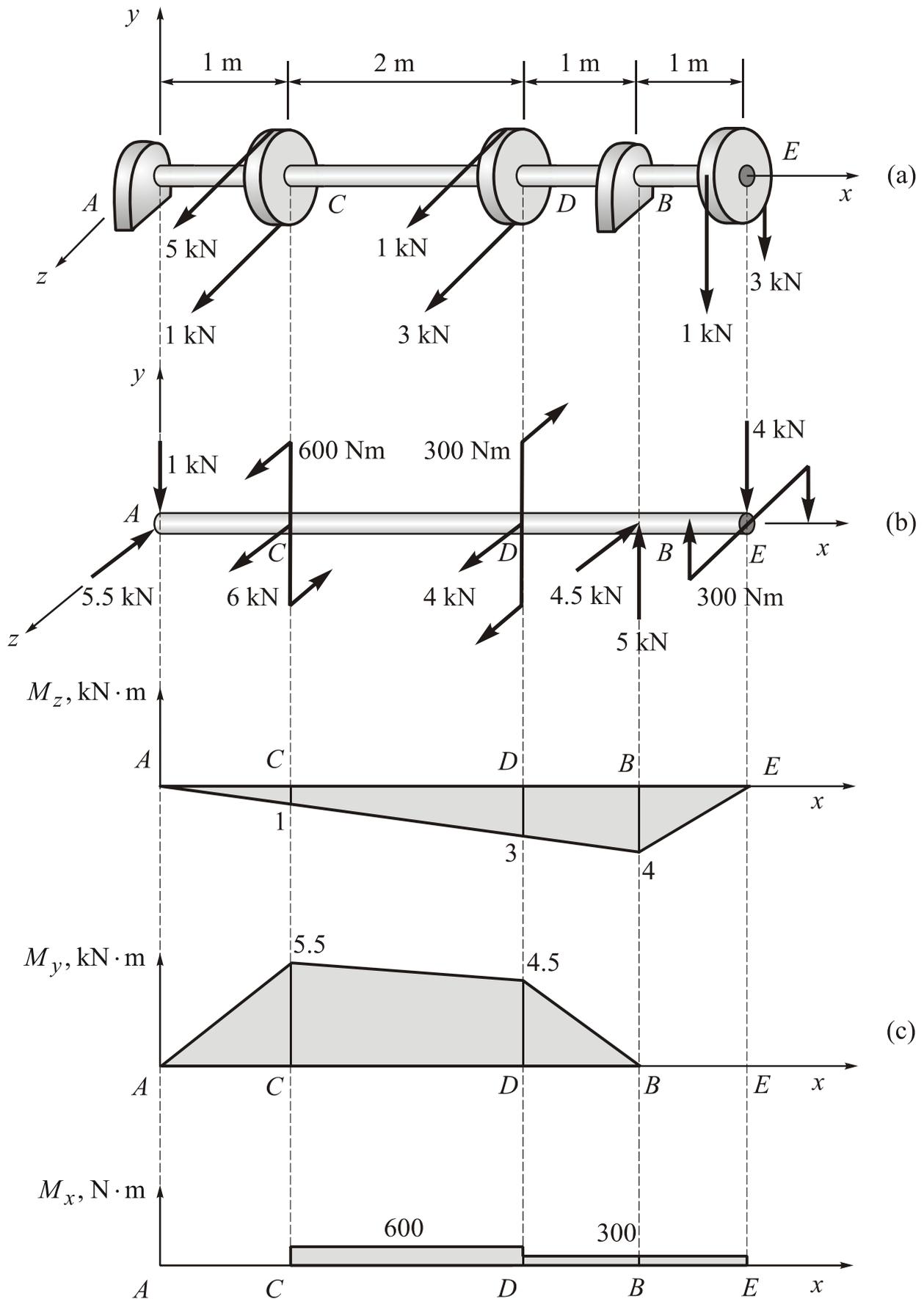
while at  $D$  and  $B$  points, we find  $5.44 \text{ kN}\cdot\text{m}$  and  $4.01 \text{ kN}\cdot\text{m}$ , respectively. Hence the critical section is at  $C$  point.

Applying the equation in accordance with the maximum shear stress theory ( $\sigma_{all} = 2\tau_{all}$ )

$$d = \sqrt[3]{\frac{16}{\pi\tau_{all}} \sqrt{M_z^2 + M_y^2 + M_x^2}}.$$

Substituting, we obtain

$$d = \sqrt[3]{\frac{16(5.62 \times 10^3)}{\pi(70 \times 10^6)}} = 74.2 \text{ mm}.$$



**Example 6.10**

A circular cast-iron shaft ( $\sigma_{ult_t} = 160$  MPa and  $\sigma_{ult_c} = 520$  MPa), rotating at 600 rpm and transmitting 40 kW is subjected to bending moment  $M_B = 400$  N·m. Using a safety factor of 2, determine the required diameter of the shaft on the basis of the maximum principal stress theory.

**Solution** Because the frequency  $f = 600$  rpm = 10 Hz, we calculate the shaft torque moment:

$$M_x = \frac{159kW}{f} = \frac{159(40)}{10} = 636 \text{ N} \cdot \text{m}.$$

The allowable stresses in tension and compression are 80 and 260 MPa, respectively.

Since the principal stresses and the maximum shearing stress for members subjected to combined normal stress in bending and shear stress in torsion are, respectively,

$$\sigma_{1,2} = \frac{\sigma}{2} \pm \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}, \quad (\text{a})$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}, \quad (\text{b})$$

setting  $\sigma_{all} = \sigma_1/f_s$ , in accordance with the maximum principal stress theory of failure, we have the following design formula:

$$d = \sqrt[3]{\frac{16}{\pi\sigma_{all}} \left( M_B + \sqrt{M_B^2 + M_x^2} \right)}. \quad (\text{c})$$

Introducing the data, we obtain

$$d = \sqrt[3]{\frac{16}{\pi(80 \times 10^6)} \left( 400 + \sqrt{400^2 + 636^2} \right)} = 41.9 \times 10^{-3} \text{ m} = 41.9 \text{ mm}$$

and

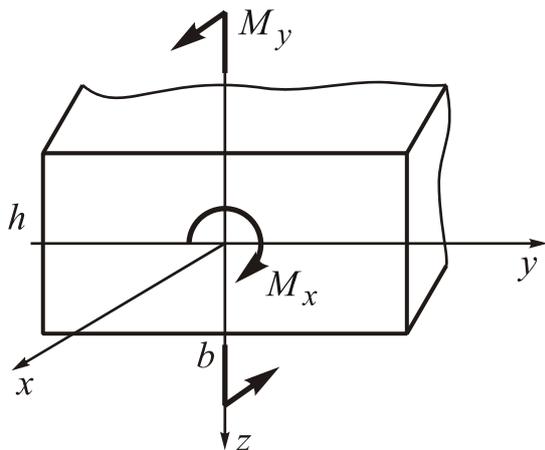
$$d = \sqrt[3]{\frac{16}{\pi(260 \times 10^6)} \left( 400 + \sqrt{400^2 + 636^2} \right)} = 28.3 \times 10^{-3} \text{ m} = 28.3 \text{ mm}.$$

Hence the allowable diameter is 41.9 mm.

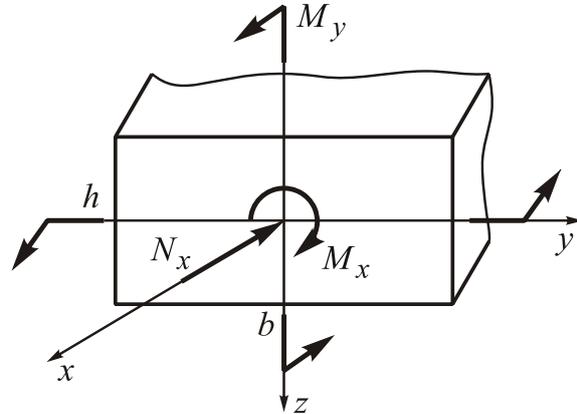


## PROBLEMS

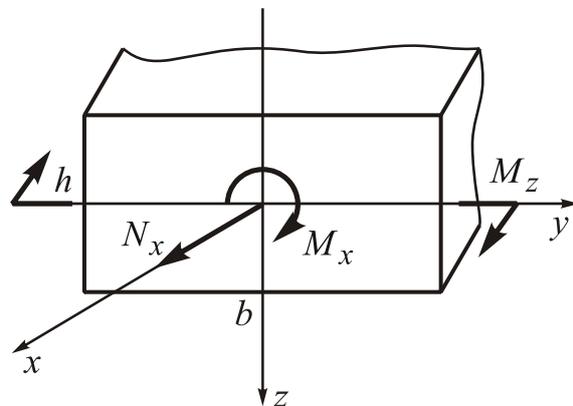
**Problem 6.1** The rectangular beam in the figure is loaded by two bending moments  $M_y = 10 \text{ kN}\cdot\text{m}$ , acting in its vertical plane of symmetry and also by torque moment  $M_x = 70 \text{ kN}\cdot\text{m}$ . Assuming the allowable stress of beam material  $\sigma_{all} = 160 \text{ MPa}$ , find cross-sectional dimensions  $b$  and  $h$ , if  $b/h = 2$ . Use maximum shear stress and maximum distortion energy theories of failure.



**Problem 6.2** Estimate the strength of rectangular beam loaded by two bending moments  $M_y = 10 \text{ kN}\cdot\text{m}$  and  $M_z = 20 \text{ kN}\cdot\text{m}$  acting respectively in vertical and horizontal planes of cross-sectional symmetry, and also by torque moment  $M_x = 15 \text{ kN}\cdot\text{m}$  and normal force  $N_x = 20 \text{ kN}$ . Assume, that  $\sigma_{all} = 160 \text{ MPa}$  and  $h = 15 \text{ cm}$ ,  $b = 30 \text{ cm}$ .

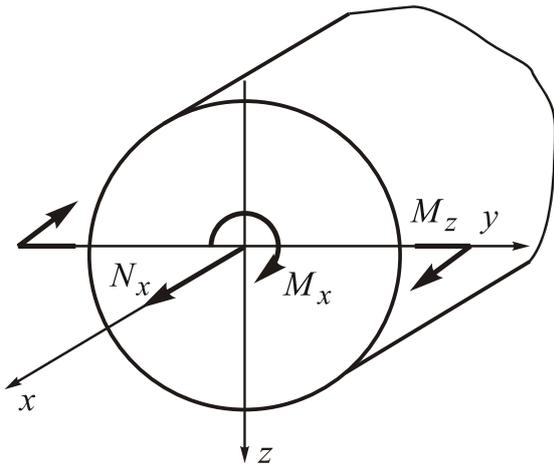


**Problem 6.3** The rectangular beam in the figure is loaded by the bending moment  $M_z = 10 \text{ kN}\cdot\text{m}$  acting in horizontal plane of cross-sectional symmetry, and also by torque moment  $M_x = 60 \text{ kN}\cdot\text{m}$  and normal force  $N_x = 60 \text{ kN}$ . Assuming the allowable stress of beam material  $\sigma_{all} = 120 \text{ MPa}$ , find cross-sectional dimensions  $h$  and  $b$  if  $b/h = 2$ . Use maximum distortion energy and maximum shear stress theories of failure.

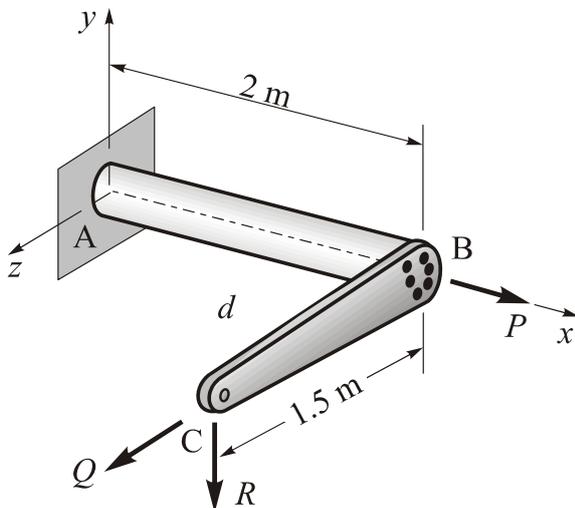


**Problem 6.4** The round solid rod in the figure is loaded by the bending moment  $M_z = 10 \text{ kN}\cdot\text{m}$  acting

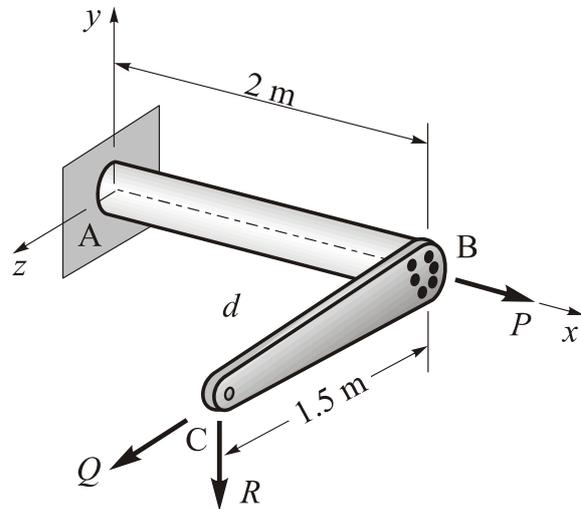
in its horizontal plane of symmetry, and also by torque moment  $M_x = 80 \text{ kN}\cdot\text{m}$  and normal force  $N_x = 100 \text{ kN}$ . Assuming that allowable stress of beam material  $\sigma_{all} = 160 \text{ MPa}$ , find cross-sectional diameter  $d$  taking into account that maximum distortion energy theory of failure is used.



**Problem 6.5** A  $d = 50\text{-mm}$ -diameter steel shaft ( $\sigma_y = 260 \text{ MPa}$ ,  $\tau_y = 140 \text{ MPa}$ ) (see figure) subjected to a load  $R$  and  $P = Q = 0$ . If the factor of safety is  $f_s = 2$ , calculate the largest permissible value of  $R$  in accordance with (a) the maximum shear stress theory and (b) the von Mises theory.



**Problem 6.6** A  $d = 50\text{-mm}$ -diameter steel shaft ( $\sigma_y = 260 \text{ MPa}$ ,  $\tau_y = 140 \text{ MPa}$ ) (see figure) subjected to a load  $R$  and  $P = 30R$ ,  $Q = 0$ . If the factor of safety is  $f_s = 2$ , calculate the largest permissible value of  $R$  in accordance with (a) the maximum shear stress theory and (b) the von Mises theory.



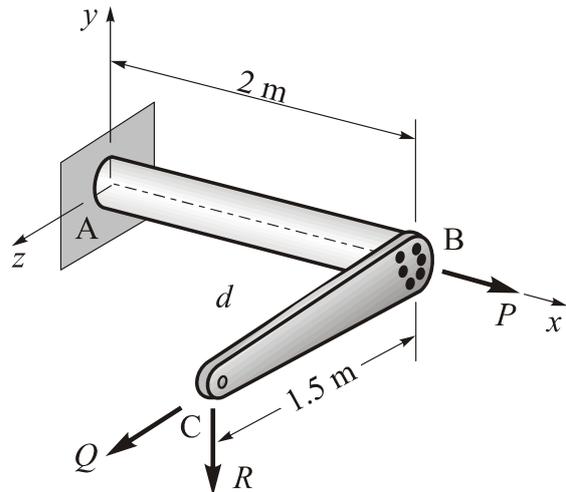
**Problem 6.7** A thin-walled, closed-ended metal tube ( $\sigma_{ult_t} = 250 \text{ MPa}$ ,  $\sigma_{ult_c} = 380 \text{ MPa}$ ) having outer and inner diameters of 20 cm and 19 cm, respectively, is subjected to an internal pressure of 5 MPa and a torque moment of 50 kN·m. Determine the factor of safety  $f_s$ , according to the maximum principal stress theory.

**Problem 6.8** A steel circular bar ( $\sigma_y = 200 \text{ MPa}$ ) of 50-mm diameter is acted upon by combined moments  $M$

and axial compressive loads  $P$  at its ends. If  $M = 2.0 \text{ kN}\cdot\text{m}$ , calculate, on the basis of the von Mises theory, the maximum permissible value of  $P$ .

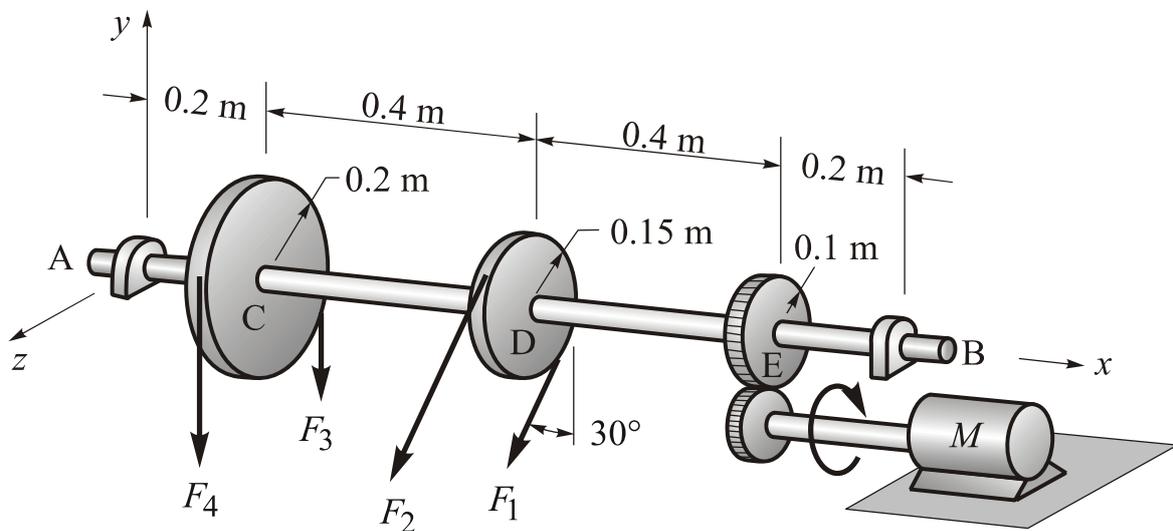
**Problem 6.9** A 40-mm-wide, 80-mm-deep, and 2.0-m-long cantilever beam is subjected to a concentrated load of  $P = 5 \text{ kN}$  at its free end. For  $\sigma_y = 240 \text{ MPa}$ , what is the factor of safety  $f_s$ ? Assume that failure occurs in accordance with the maximum energy of distortion theory.

**Problem 6.10** Design a solid steel shaft ( $\tau_y = 60 \text{ MPa}$ ) subjected to loads of  $R = 500 \text{ N}$ ,  $Q = 300 \text{ N}$ , and  $P = 0$ . Apply the maximum shear stress theory using a factor of safety  $f_s = 1.2$ .

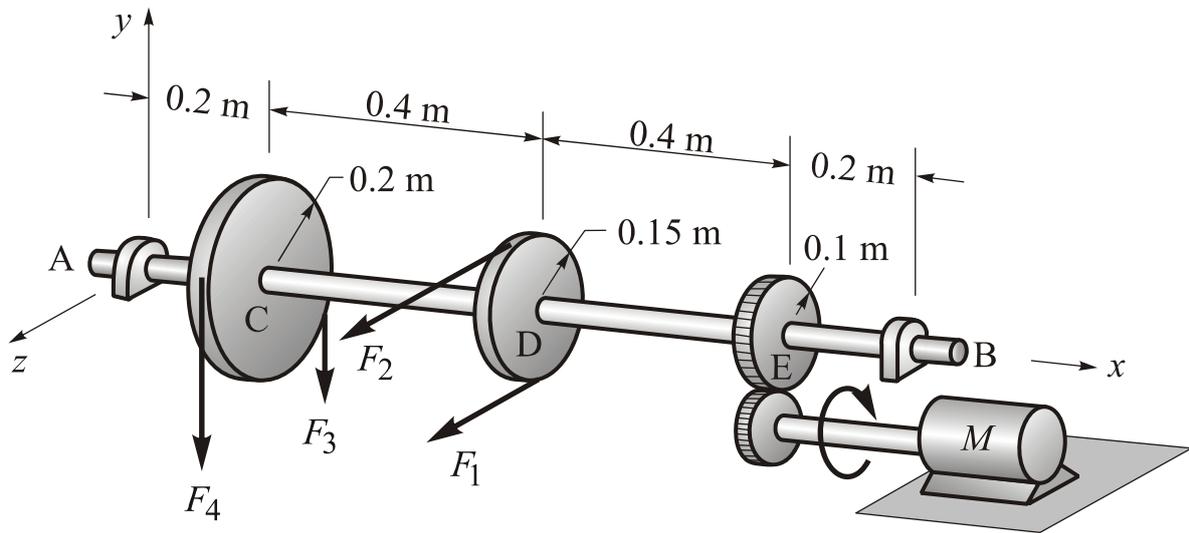


**Problem 6.11** A 6-m-long steel shaft of allowable strength  $\sigma_{all} = 120 \text{ MPa}$  carries a torque moment of  $500 \text{ N}\cdot\text{m}$  and its own weight. Use  $\rho = 7.86 \text{ Mg/m}^3$  as the mass per unit volume and assume that the shaft is supported by frictionless bearings at its ends. Calculate the required shaft diameter in accordance with the von Mises theory of failure.

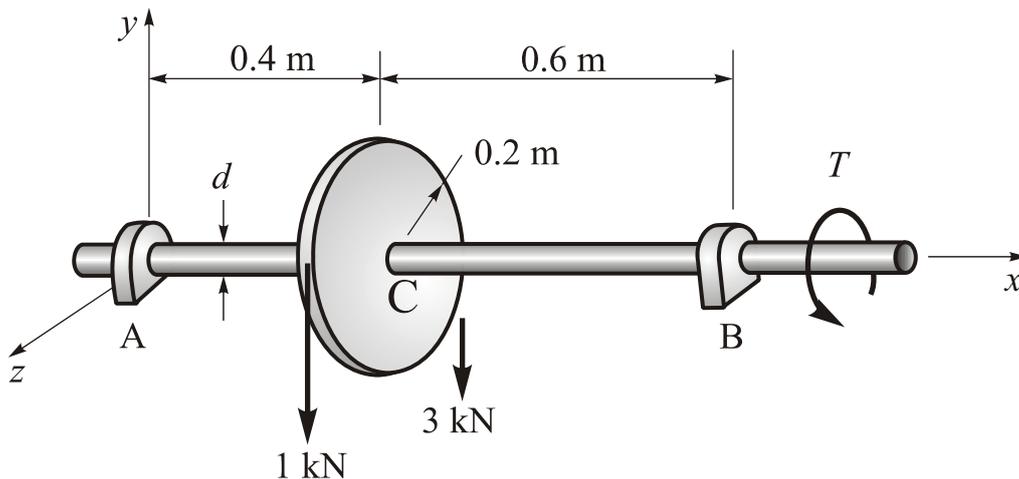
**Problem 6.12** A solid shaft  $AB$  is to transmit  $20 \text{ kW}$  at  $180 \text{ rpm}$  from the motor and gear to pulley  $D$ , where  $8 \text{ kW}$  is taken off, and to pulley  $C$  where the remaining  $12 \text{ kW}$  is taken off (see figure). Assume the ratios of the pulley tensions to be  $F_1/F_2 = 3$  and  $F_3/F_4 = 3$ . If  $\tau_{all} = 50 \text{ MPa}$ , determine the required diameter of the shaft  $AB$  according to the maximum shear stress theory of failure.



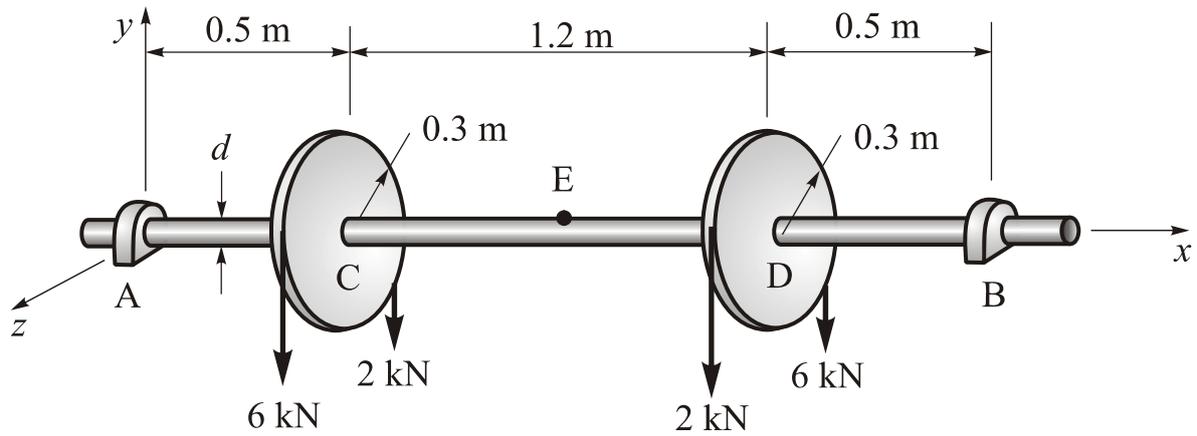
**Problem 6.13** A solid shaft  $AB$  is to transmit 20 kW at 180 rpm from the motor and gear to pulley  $D$ , where 8 kW is taken off, and to pulley  $C$  where the remaining 12 kW is taken off. Assume the ratios of the pulley tensions to be  $F_1/F_2 = 3$  and  $F_3/F_4 = 3$ . If  $\tau_{all} = 60$  MPa, determine the required diameter of the shaft  $AB$  according to the maximum shear stress theory of failure.



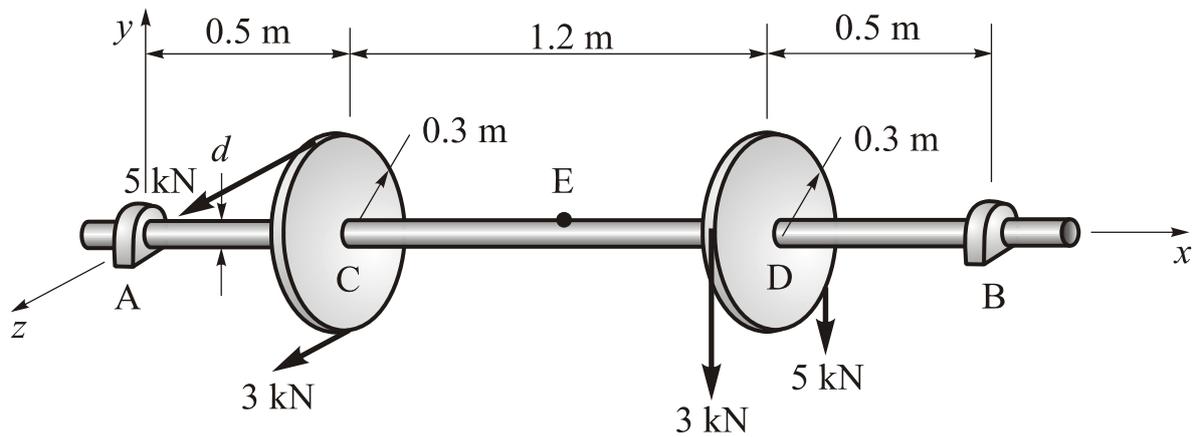
**Problem 6.14** Design the steel shaft ( $\tau_y = 60$  MPa) of the system shown in the figure. Use the maximum shear stress theory of failure and a factor of safety  $f_s = 1.4$ .



**Problem 6.15** For the shaft of the system shown in the figure, determine the diameter  $d$ , applying the maximum principal stress theory of failure. Use  $\sigma_{all} = 160$  MPa.



**Problem 6.16** For the shaft of the system shown in the figure, determine the diameter  $d$ , applying the maximum principal stress theory of failure. Use  $\sigma_{all} = 160$  MPa.



## Chapter 7 Appendixes

### Appendix A Properties of Selected Engineering Materials

#### A.1 Density

**Table A.1 Density Values for Various Engineering Materials (Room-Temperature Conditions)**

Density	
Material	$10^3 \text{ kg/m}^3$
1	2
<b>METALS AND METAL ALLOYS</b>	
<b>Plain Carbon and Low Alloy Steels</b>	
Steel alloy A36	7.85
Steel alloy 1020	7.85
Steel alloy 1040	7.85
Steel alloy 4140	7.85
Steel alloy 4340	7.85
<b>Stainless Steels</b>	
Stainless alloy 304	8.00
Stainless alloy 316	8.00
Stainless alloy 405	7.80
Stainless alloy 440A	7.80
Stainless alloy 17-7PH	7.65
<b>Cast Irons</b>	
Gray irons	
• Grade G1800	7.30
• Grade G3000	7.30
• Grade G4000	7.30
Ductile irons	
• Grade 60-40-18	7.10
• Grade 80-55-06	7.10
• Grade 120-90-02	7.10
<b>Aluminum Alloys</b>	
Alloy 1100	2.71
Alloy 2024	2.77
Alloy 6061	2.70
Alloy 7075	2.80
Alloy 356.0	2.69
<b>Copper Alloys</b>	
C11000 (electrolytic tough pitch)	8.89
C17200 (beryllium-copper)	8.25
C26000 (cartridge brass)	8.53
C36000 (free-cutting brass)	8.50
C71500 (copper-nickel, 30%)	8.94
C93200 (bearing bronze)	8.93
<b>Magnesium Alloys</b>	
Alloy AZ31B	1.77
Alloy AZ91D	1.81
<b>Titanium Alloys</b>	
Commercially pure (ASTM grade 1)	4.51
Alloy Ti-5Al-2.5Sn	4.48
Alloy Ti-6Al-4V	4.43

Table A.1 (continued)

1	2
<b>Precious Metals</b>	
Gold (commercially pure)	19.32
Platinum (commercially pure)	21.45
Silver (commercially pure)	10.49
<b>Refractory Metals</b>	
Molybdenum (commercially pure)	10.22
Tantalum (commercially pure)	16.6
Tungsten (commercially pure)	19.3
<b>Miscellaneous Nonferrous Alloys</b>	
Nickel 200	8.89
Inconel 625	8.44
Monel 400	8.80
Haynes alloy 25	9.13
Invar	8.05
Super invar	8.10
Kovar	8.36
Chemical lead	11.34
Antimonial lead (6%)	10.88
Tin (commercially pure)	7.17
Lead-Tin solder (60Sn-40Pb)	8.52
Zinc (commercially pure)	7.14
Zirconium, reactor grade 702	6.51
<b>GRAPHITE, CERAMICS, AND SEMICONDUCTING MATERIALS</b>	
Aluminum oxide	
• 99.9% pure	3.98
• 96%	3.72
• 90%	3.60
Concrete	2.4
Diamond	
• Natural	3.51
• Synthetic	3.20-3.52
Gallium arsenide	5.32
Glass, borosilicate (Pyrex)	2.23
Glass, soda-lime	2.5
Glass ceramic (Pyroceram)	2.60
Graphite	
• Extruded	1.71
• Isostatically molded	1.78
Silica, fused	2.2
Silicon	2.33
Silicon carbide	
• Hot pressed	3.3
• Sintered	3.2
Silicon nitride	
• Hot pressed	3.3
• Reaction bonded	2.7
• Sintered	3.3
Zirconia, 3 mol% Y <sub>2</sub> O <sub>3</sub> , sintered	6.0

Table A.1 (finished)

1	2
<b>POLYMERS</b>	
Elastomers	
• Butadiene-acrylonitrile (nitrile)	0.98
• Styrene-butadiene (SBR)	0.94
• Silicone	1.1-1.6
Epoxy	1.11-1.40
Nylon 6,6	1.14
Phenolic	1.28
Polybutylene terephthalate (PBT)	1.34
Polycarbonate (PC)	1.20
Polyester (thermoset)	1.04-1.46
Polyetheretherketone (PEEK)	1.31
Polyethylene	
• Low density (LDPE)	0.925
• High density (HDPE)	0.959
• Ultrahigh molecular weight (UHMWPE)	0.94
Polyethylene terephthalate (PET)	1.35
Polymethyl methacrylate (PMMA)	1.19
Polypropylene (PP)	0.905
Polystyrene (PS)	1.05
Polytetrafluoroethylene (PTFE)	2.17
Polyvinyl chloride (PVC)	1.30-1.58
<b>FIBER MATERIALS</b>	
Aramid (Kevlar 49)	1.44
Carbon (PAN precursor)	
• Standard modulus	1.78
• Intermediate modulus	1.78
• High modulus	1.81
E Glass	2.58
<b>COMPOSITE MATERIALS</b>	
Aramid fibers-epoxy matrix ( $V_f = 0.60$ )	1.4
High modulus carbon fibers-epoxy matrix ( $V_f = 0.60$ )	1.7
E glass fibers-epoxy matrix ( $V_f = 0.60$ )	2.1
Wood	
• Douglas fir (12% moisture)	0.46-0.50
• Red oak (12% moisture)	0.61-0.67

## A.2 Modulus of Elasticity

**Table A.2 Modulus of Elasticity Values for Various Engineering Materials (Room-Temperature Conditions)**

Material	Modulus of Elasticity	
	GPa	10 <sup>6</sup> psi
1	2	3
<b>METALS AND METAL ALLOYS</b>		
<b>Plain Carbon and Low Alloy Steels</b>		
Steel alloy A36	207	30
Steel alloy 1020	207	30
Steel alloy 1040	207	30
Steel alloy 4140	207	30
Steel alloy 4340	207	30
Stainless alloy 304	193	28
Stainless alloy 316	193	28
Stainless alloy 405	200	29
Stainless alloy 440A	200	29
Stainless alloy 17-7PH	204	29.5
<b>Cast Irons</b>		
Gray irons		
• Grade G1800	66-97 <sup>a</sup>	9.6-14 <sup>a</sup>
• Grade G3000	90-113 <sup>a</sup>	13.0-16.4 <sup>a</sup>
• Grade G4000	110-138 <sup>a</sup>	16-20 <sup>a</sup>
Ductile irons		
• Grade 60-40-18	169	24.5
• Grade 80-55-06	168	24.4
• Grade 120-90-02	164	23.8
<b>Aluminum Alloys</b>		
Alloy 1100	69	10
Alloy 2024	72.4	10.5
Alloy 6061	69	10
Alloy 7075	71	10.3
Alloy 356.0	72.4	10.5
<b>Copper Alloys</b>		
C11000 (electrolytic tough pitch)	115	16.7
C17200 (beryllium-copper)	128	18.6
C26000 (cartridge brass)	110	16
C36000 (free-cutting brass)	97	14
C71500 (copper-nickel, 30%)	150	21.8
C93200 (bearing bronze)	100	14.5

Table A.2 (continued)

1	2	3
<b>Magnesium Alloys</b>		
Alloy AZ31B	45	6.5
Alloy AZ91D	45	6.5
<b>Titanium Alloys</b>		
Commercially pure (ASTM grade 1)	103	14.9
Alloy Ti-5Al-2.5Sn	110	16
Alloy Ti-6Al-4V	114	16.5
<b>Precious Metals</b>		
Gold (commercially pure)	77	11.2
Platinum (commercially pure)	171	24.8
Silver (commercially pure)	74	10.7
<b>Refractory Metals</b>		
Molybdenum (commercially pure)	320	46.4
Tantalum (commercially pure)	185	27
Tungsten (commercially pure)	400	58
<b>Miscellaneous Nonferrous Alloys</b>		
Nickel 200	204	29.6
Inconel 625	207	30
Monel 400	180	26
Haynes alloy 25	236	34.2
Invar	141	20.5
Super invar	144	21
Kovar	207	30
Chemical lead	13.5	2
Tin (commercially pure)	44.3	6.4
Lead-Tin solder (60Sn-40Pb)	30	4.4
Zinc (commercially pure)	104.5	15.2
Zirconium, reactor grade 702	99.3	14.4
<b>GRAPHITE, CERAMICS, AND SEMICONDUCTING MATERIALS</b>		
Aluminum oxide		
• 99.9% pure	380	55
• 96%	303	44
• 90%	275	40
Concrete	25.4-36.6 <sup>a</sup>	3.7-5.3 <sup>a</sup>
Diamond		
• Natural	700-1200	102-174
• Synthetic	800-925	116-134

Table A.2 (continued)

1	2	3
Gallium arsenide, single crystal	85	12.3
• In the (100) direction		
• In the (110) direction	122	17.7
• In the (111) direction	142	20.6
Glass, borosilicate (Pyrex)	70	10.1
Glass, soda-lime	69	10
Glass ceramic (Pyroceram)	120	17.4
Graphite		
• Extruded	11	1.6
• Isostatically molded	11.7	1.7
Silica, fused	73	10.6
Silicon, single crystal		
• In the (100) direction	129	18.7
• In the (110) direction	168	24.4
• In the (111) direction	187	27.1
Silicon carbide		
• Hot pressed	207-483	30-70
• Sintered	207-483	30-70
Silicon nitride		
• Hot pressed	304	44.1
• Reaction bonded	304	44.1
• Sintered	304	44.1
Zirconia, 3 mol% Y <sub>2</sub> O <sub>3</sub>	205	30
<b>POLYMERS</b>		
Elastomers	0.0034 <sup>b</sup>	0.00049 <sup>b</sup>
• Butadiene-acrylonitrile (nitrile)		
• Styrene-butadiene (SBR)	0.002-0.010 <sup>b</sup>	0.0003-0.0015 <sup>b</sup>
Epoxy	2.41	0.35
Nylon 6,6	1.59-3.79	0.230-0.550
Phenolic	2.76-4.83	0.40-0.70
Polybutylene terephthalate (PBT)	1.93-3.00	0.280-0.435
Polycarbonate (PC)	2.38	0.345
Polyester (thermoset)	2.06-4.41	0.30-0.64
Polyetheretherketone (PEEK)	1.10	0.16
Polyethylene		
• Low density (LDPE)	0.172-0.282	0.025-0.041
• High density (HDPE)	1.08	0.157
• Ultrahigh molecular weight (UHMWPE)	0.69	0.100

Table A.2 (finished)

1	2	3
Polyethylene terephthalate (PET)	2.76-4.14	0.40-0.60
Polymethyl methacrylate (PMMA)	2.24-3.24	0.325-0.470
Polypropylene (PP)	1.14-1.55	0.165-0.225
Polystyrene (PS)	2.28-3.28	0.330-0.475
Polytetrafluoroethylene (PTFE)	0.40-0.55	0.058-0.080
Polyvinyl chloride (PVC)	2.41-4.14	0.35-0.60
<b>FIBER MATERIALS</b>		
Aramid (Kevlar 49)	131	19
Carbon (PAN precursor)		
• Standard modulus	230	33.4
• Intermediate modulus	285	41.3
• High modulus	400	58
E Glass	72.5	10.5
<b>COMPOSITE MATERIALS</b>		
Aramid fibers-epoxy matrix ( $V_f = 0.60$ )		
Longitudinal	76	11
Transverse	5.5	0.8
High modulus carbon fibers-epoxy matrix ( $V_f = 0.60$ )		
Longitudinal	220	32
Transverse	6.9	1.0
E glass fibers-epoxy matrix ( $V_f = 0.60$ )		
Longitudinal	45	6.5
Transverse	12	1.8
Wood		
• Douglas fir (12% moisture)		
Parallel to grain	10.8-13.6 <sup>c</sup>	1.57-1.97 <sup>c</sup>
Perpendicular to grain	0.54-0.68 <sup>c</sup>	0.078-0.10 <sup>c</sup>
• Red oak (12% moisture)		
Parallel to grain	11.0-14.1 <sup>c</sup>	1.60-2.04 <sup>c</sup>
Perpendicular to grain	0.55-0.71 <sup>c</sup>	0.08-0.10 <sup>c</sup>

<sup>a</sup> Secant modulus taken at 25% of ultimate strength.

<sup>b</sup> Modulus taken at 100% elongation.

<sup>c</sup> Measured in bending.

## A.3 Poisson's Ratio

**Table A.3 Poisson's Ratio Values for Various Engineering Materials (Room-Temperature Conditions)**

Material	Poisson's Ratio	Material	Poisson's Ratio
1	2	3	4
<b>METALS AND METAL ALLOY</b>		<b>Refractory Metals</b>	
<b>Plain Carbon and Low Alloy Steels</b>		Molybdenum (commercially pure)	0.32
Steel alloy A36	0.30	Tantalum (commercially pure)	0.35
Steel alloy 1020	0.30	Tungsten (commercially pure)	0.28
Steel alloy 1040	0.30		
Steel alloy 4140	0.30		
Steel alloy 4340	0.30		
<b>Stainless Steels</b>		<b>Miscellaneous Nonferrous Alloys</b>	
Stainless alloy 304	0.30	Nickel 200	0.31
Stainless alloy 316	0.30	Inconel 625	0.31
Stainless alloy 405	0.30	Monel 400	0.32
Stainless alloy 440A	0.30	Chemical lead	0.44
Stainless alloy 17-7PH	0.30	Tin (commercially pure)	0.33
<b>Cast Irons</b>		Zinc (commercially pure)	0.25
Gray irons		Zirconium, reactor grade 702	0.35
• Grade G1800	0.26	<b>GRAPHITE, CERAMICS, AND SEMICONDUCTING MATERIALS</b>	
• Grade G3000	0.26	Aluminum oxide	
• Grade G4000	0.26	• 99.9% pure	0.22
Ductile irons		• 96%	0.21
• Grade 60-40-18	0.29	• 90%	0.22
• Grade 80-55-06	0.31	Concrete	0.20
• Grade 120-90-02	0.28	Diamond	
<b>Aluminum Alloys</b>		• Natural	0.10-0.30
Alloy 1100	0.33	• Synthetic	0.20
Alloy 2024	0.33	Gallium arsenide	
Alloy 6061	0.33	• (100) orientation	0.30
Alloy 7075	0.33	Glass, borosilicate (Pyrex)	0.20
Alloy 356.0	0.33	Glass, soda-lime	0.23
<b>Copper Alloys</b>		Glass ceramic (Pyroceram)	0.25
C11000 (electrolytic tough pitch)	0.33	Silica, fused	0.17
C17200 (beryllium-copper)	0.30		

Table A.3 (finished)

1	2	3	4
C26000 (cartridge brass)	0.35	Silicon	
C36000 (free-cutting brass)	0.34	• (100) orientation	0.28
C71500 (copper-nickel, 30%)	0.34	• (111) orientation	0.36
C93200 (bearing bronze)	0.34	Silicon carbide	
<b>Magnesium Alloys</b>		• Hot pressed	0.17
Alloy AZ31B	0.35	• Sintered	0.16
Alloy AZ91D	0.35	Silicon nitride	
<b>Titanium Alloys</b>		• Hot pressed	0.30
Commercially pure (ASTM grade 1)	0.34	• Reaction bonded	0.22
Alloy Ti-5Al-2.5Sn	0.34	• Sintered	0.28
Alloy Ti-6Al-4V	0.34	Zirconia, 3 mol% Y <sub>2</sub> O <sub>3</sub>	0.31
		<b>POLYMERS</b>	
<b>Precious Metals</b>		Nylon 6,6	0.39
Gold (commercially pure)	0.42	Polycarbonate (PC)	0.36
Platinum (commercially pure)	0.39	Polystyrene (PS)	0.33
		Polytetrafluoroethylene (PTFE)	0.46
Silver (commercially pure)	0.37		
<b>COMPOSITE MATERIALS</b>		Polyvinyl chloride (PVC)	0.38
Aramid fibers-epoxy matrix ( $V_f = 0.6$ )	0.34	<b>FIBER MATERIALS</b>	
		E Glass	0.22
High modulus carbon fibers-epoxy matrix ( $V_f = 0.6$ )	0.25	E glass fibers-epoxy matrix ( $V_f = 0.6$ )	0.19

**Table A.4 Elastic and Shear Moduli, and Poisson's Ratio for Main Classes of Materials (Room-Temperature Conditions)**

Material	Modulus of Elasticity		Shear Modulus		Poisson's Ratio
	GPa	10 <sup>6</sup> psi	GPa	10 <sup>6</sup> psi	
1	2	3	4	5	6
<b>Metal Alloys</b>					
Tungsten	407	59	160	23.2	0.28
Steel	207	30	83	12.0	0.30
Nickel	207	30	76	11.0	0.31
Titanium	107	15.5	45	6.5	0.34
Copper	110	16	46	6.7	0.34

Table A.4 (finished)

1	2	3	4	5	6
Brass	97	14	37	5.4	0.34
Aluminum	69	10	25	3.6	0.33
Magnesium	45	6.5	17	2.5	0.35
<b>Ceramic Materials</b>					
Aluminum oxide (Al <sub>2</sub> O <sub>3</sub> )	393	57	—	—	0.22
Silicon carbide (SiC)	345	50	—	—	0.17
Silicon nitride (Si <sub>3</sub> N <sub>4</sub> )	304	44	—	—	0.30
Spinel (MgAl <sub>2</sub> O <sub>4</sub> )	260	38	—	—	—
Magnesium oxide (MgO)	225	33	—	—	0.18
Zirconia <sup>a</sup>	205	30	—	—	0.31
Mullite (3Al <sub>2</sub> O <sub>3</sub> -2SiO <sub>2</sub> )	145	21	—	—	0.24
Glass-ceramic (Pyroceram)	120	17	—	—	0.25
Fused silica (SiO <sub>2</sub> )	73	11	—	—	0.17
Soda-lime glass	69	10	—	—	0.23
<b>Polymers</b>					
Phenol-formaldehyde	2.76-4.83	0.40-0.70	—	—	—
Polyvinyl chloride (PVC)	2.41-4.14	0.35-0.60	—	—	0.38
Polyester (PET)	2.76-4.14	0.40-0.60	—	—	—
Polystyrene (PS)	2.28-3.28	0.33-0.48	—	—	0.33
Polymethyl methacrylate (PMMA)	24-3.24	0.33-0.47	—	—	—
Polycarbonate (PC)	2.38	0.35	—	—	0.36
Nylon 6,6	1.58-3.80	0.23-0.55	—	—	0.39
Polypropylene (PP)	1.14-1.55	0.17-0.23	—	—	—
Polyethylene – high density (HDPE)	1.08	0.16	—	—	—
Polytetrafluoroethylene (PTFE)	0.40-0.55	0.058-0.080	—	—	0.46
Polyethylene – low density (LDPE)	0.17-0.28	0.025-0.041	—	—	—

<sup>a</sup> Partially stabilized with 3 mol% Y<sub>2</sub>O<sub>3</sub>.

## A.4 Strength and Ductility

**Table A.5 Yield Strength, Tensile Strength, and Ductility (Percent Elongation) Values for Main Classes of Engineering Materials (Room-Temperature Conditions)**

Material/ Condition	Yield Strength (MPa [ksi])	Tensile Strength (MPa [ksi])	Percent Elongation
1	2	3	4
<b>METALS AND METAL ALLOYS</b>			
<b>Plain Carbon and Low Alloy Steels</b>			
Steel alloy A36 • Hot rolled	220-250 (32-36)	400-500 (58-72.5)	23
Steel alloy 1020 • Hot rolled	210 (30) (min)	380 (55) (min)	25 (min)
• Cold drawn	350 (51) (min)	420 (61) (min)	15 (min)
• Annealed (@ 870°C)	295 (42.8)	395 (57.3)	36.5
• Normalized (@ 925°C)	345 (50.3)	440 (64)	38.5
Steel alloy 1040 • Hot rolled	290 (42) (min)	520 (76) (min)	18 (min)
• Cold drawn	490 (71) (min)	590 (85) (min)	12 (min)
• Annealed (@ 785°C)	355 (51.3)	520 (75.3)	30.2
• Normalized (@ 900°C)	375 (54.3)	590 (85)	28.0
Steel alloy 4140 • Annealed (@ 815°C)	417 (60.5)	655 (95)	25.7
• Normalized (@ 870°C)	655 (95)	1020 (148)	17.7
• Oil-quenched and tempered (@ 315°C)	1570 (228)	1720 (250)	11.5
Steel alloy 4340 • Annealed (@ 810°C)	472 (68.5)	745 (108)	22
• Normalized (@ 870°C)	862 (125)	1280 (185.5)	12.2
• Oil-quenched and tempered (@ 315°C)	1620 (235)	1760 (255)	12
<b>Stainless Steels</b>			
Stainless alloy 304 • Hot finished and annealed	205 (30) (min)	515 (75) (min)	40 (min)
• Cold worked (1/4 hard)	515 (75) (min)	860 (125) (min)	10 (min)
Stainless alloy 316 • Hot finished and annealed	205 (30) (min)	515 (75) (min)	40 (min)
• Cold drawn and annealed	310 (45) (min)	620 (90) (min)	30 (min)
Stainless alloy 405 • Annealed	170 (25)	415 (60)	20
Stainless alloy 440A • Annealed	415 (60)	725 (105)	20
• Tempered @ 315°C	1650 (240)	1790 (260)	5
Stainless alloy 17-7PH • Cold rolled	1210 (175) (min)	1380 (200) (min)	1 (min)
• Precipitation hardened @ 510°C	1310 (190) (min)	1450 (210) (min)	3.5 (min)

Table A.5 (continued)

1	2	3	4
<b>Cast Irons</b>			
Gray irons			
• Grade G1800 (as cast)	—	124 (18) (min)	—
• Grade G3000 (as cast)	—	207 (30) (min)	—
• Grade G4000 (as cast)	—	276 (40) (min)	—
Ductile irons			
• Grade 60-40-18 (annealed)	276 (40) (min)	414 (60) (min)	18 (min)
• Grade 80-55-06 (as cast)	379 (55) (min)	552 (80) (min)	6 (min)
• Grade 120-90-02 (oil quenched and tempered)	621 (90) (min)	827 (120) (min)	2 (min)
<b>Aluminum Alloys</b>			
Alloy 1100			
• Annealed (O temper)	34(5)	90 (13)	40
• Strain hardened (H14 temper)	117 (17)	124 (18)	15
Alloy 2024	75 (11)	185 (27)	20
• Annealed (O temper)			
• Heat treated and aged (T3 temper)	345 (50)	485 (70)	18
• Heat treated and aged (T351 temper)	325 (47)	470 (68)	20
Alloy 6061			
• Annealed (O temper)	55(8)	124 (18)	30
• Heat treated and aged (T6 and T651 tempers)	276 (40)	310 (45)	17
Alloy 7075			
• Annealed (O temper)	103 (15)	228 (33)	17
• Heat treated and aged (T6 temper)	505 (73)	572 (83)	11
Alloy 356.0			
• As cast	124 (18)	164 (24)	6
• Heat treated and aged (T6 temper)	164 (24)	228 (33)	3.5
<b>Copper Alloys</b>			
C11000 (electrolytic tough pitch)			
• Hot rolled	69 (10)	220 (32)	50
• Cold worked (H04 temper)	310 (45)	345 (50)	12
C17200 (beryllium-copper)			
• Solution heat treated	195-380 (28-55)	415-540 (60-78)	35-60
• Solution heat treated, aged @ 330°C	965-1205 (140-175)	1140-1310 (165-190)	4-10
C26000 (cartridge brass)			
• Annealed	75-150 (11-22)	300-365 (43.5-53.0)	54-68
• Cold worked (H04 temper)	435 (63)	525 (76)	8
C36000 (free-cutting brass)			
• Annealed	125 (18)	340 (49)	53
• Cold worked (H02 temper)	310 (45)	400 (58)	25
C71500 (copper-nickel, 30%)			

Table A.5 (continued)

1	2	3	4
<ul style="list-style-type: none"> <li>• Hot rolled</li> <li>• Cold worked (H80 temper)</li> </ul> C93200 (bearing bronze)	140 (20) 545 (79) 125 (18)	380 (55) 580 (84) 240 (35)	45 3 20
<b>Magnesium Alloys</b>			
Alloy AZ31B			
<ul style="list-style-type: none"> <li>• Rolled</li> <li>• Extruded</li> </ul>	220 (32) 200 (29)	290 (42) 262 (38)	15 15
Alloy AZ91D			
<ul style="list-style-type: none"> <li>• As cast</li> </ul>	97-150 (14-22)	165-230 (24-33)	3
<b>Titanium Alloys</b>			
Commercially pure (ASTM grade 1)			
<ul style="list-style-type: none"> <li>• Annealed</li> </ul>	170 (25) (min)	240 (35) (min)	30
Alloy Ti-5Al-2.5Sn			
<ul style="list-style-type: none"> <li>• Annealed</li> </ul>	760 (110) (min)	790 (115) (min)	16
Alloy Ti-6Al-4V			
<ul style="list-style-type: none"> <li>• Annealed</li> <li>• Solution heat treated and aged</li> </ul>	830 (120) (min) 1103 (160)	900 (130) (min) 1172 (170)	14 10
<b>Precious Metals</b>			
Gold (commercially pure)			
<ul style="list-style-type: none"> <li>• Annealed</li> <li>• Cold worked (60% reduction)</li> </ul>	nil 205 (30)	130 (19) 220 (32)	45 4
Platinum (commercially pure)			
<ul style="list-style-type: none"> <li>• Annealed</li> <li>• Cold worked (50%)</li> </ul>	<13.8 (2) —	125-165 (18-24) 205-240 (30-35)	30-40 1-3
Silver (commercially pure)			
<ul style="list-style-type: none"> <li>• Annealed</li> <li>• Cold worked (50%)</li> </ul>	— —	170 (24.6) 296 (43)	44 3.5
<b>Refractory Metals</b>			
Molybdenum (commercially pure)	500 (72.5)	630 (91)	25
Tantalum (commercially pure)	165 (24)	205 (30)	40
Tungsten (commercially pure)	760 (110)	960 (139)	2
<b>Miscellaneous Nonferrous Alloys</b>			
Nickel 200 (annealed)	148 (21.5)	462 (67)	47
Inconel 625 (annealed)	517 (75)	930 (135)	42.5
Monel 400 (annealed)	240 (35)	550 (80)	40
Haynes alloy 25	445 (65)	970 (141)	62
Invar (annealed)	276 (40)	517 (75)	30
Super invar (annealed)	276 (40)	483 (70)	30
Kovar (annealed)	276 (40)	517 (75)	30
Chemical lead	6-8 (0.9-1.2)	16-19 (2.3-2.7)	30-60
Antimonial lead (6%) (chill cast)	—	47.2 (6.8)	24
Tin (commercially pure)	11 (1.6)	—	57
Lead-Tin solder (60Sn-40Pb)	—	52.5 (7.6)	30-60

Table A.5 (continued)

1	2	3	4
Zinc (commercially pure)			
• Hot rolled (anisotropic)	—	134-159 (19.4-23.0)	50-65
• Cold rolled (anisotropic)	—	145-186 (21-27)	40-50
Zirconium, reactor grade 702			
• Cold worked and annealed	207 (30) (min)	379 (55) (min)	16 (min)
<b>GRAPHITE, CERAMICS, AND SEMICONDUCTING MATERIALS<sup>a</sup></b>			
Aluminum oxide			
• 99.9% pure	—	282-551 (41-80)	—
• 96%	—	358 (52)	—
• 90%	—	337 (49)	—
Concrete <sup>b</sup>	—	37.3-41.3 (5.4-6.0)	—
Diamond			
• Natural	—	1050 (152)	—
• Synthetic	—	800-1400 (116-203)	—
Gallium arsenide			
• {100} orientation, polished surface	—	66 (9.6) <sup>c</sup>	—
• {100} orientation, as-cut surface	—	57 (8.3) <sup>c</sup>	—
Glass, borosilicate (Pyrex)	—	69 (10)	—
Glass, soda-lime	—	69 (10)	—
Glass ceramic (Pyroceram)	—	123-370 (18-54)	—
Graphite			
• Extruded (with the grain direction)	—	13.8-34.5 (2.0-5.0)	—
• Isostatically molded	—	31-69 (4.5-10)	—
Silica, fused	—	104 (15)	—
Silicon			
• {100} orientation, as-cut surface	—	130 (18.9)	—
• {100} orientation, laser scribed	—	81.8 (11.9)	—
Silicon carbide			
• Hot pressed	—	230-825 (33-120)	—
• Sintered	—	96-520 (14-75)	—
Silicon nitride			
• Hot pressed	—	700-1000 (100-150)	—
• Reaction bonded	—	250-345 (36-50)	—
• Sintered	—	414-650 (60-94)	—
Zirconia, 3 mol% Y <sub>2</sub> O <sub>3</sub> (sintered)	—	800-1500 (116-218)	—
<b>POLYMERS</b>			
Elastomers			
• Butadiene-acrylonitrile (nitrile)	—	6.9-24.1 (1.0-3.5)	400-600
• Styrene-butadiene (SBR)	—	12.4-20.7 (1.8-3.0)	450-500
• Silicone	—	10.3 (1.5)	100-800
Epoxy	—	27.6-90.0 (4.0-13)	3-6
Nylon 6,6			
• Dry, as molded	55.1-82.8 (8-12)	94.5 (13.7)	15-80
• 50% relative humidity	44.8-58.6 (6.5-8.5)	75.9 (11)	150-300
Phenolic	—	34.5-62.1 (5.0-9.0)	1.5-2.0

Table A.5 (finished)

1	2	3	4
Polybutylene terephthalate (PBT)	56.6-60.0 (8.2-8.7)	56.6-60.0 (8.2-8.7)	50-300
Polycarbonate (PC)	62.1 (9)	62.8-72.4 (9.1-10.5)	110-150
Polyester (thermoset)	—	41.4-89.7 (6.0-13.0)	<2.6
Polyetheretherketone (PEEK)	91 (13.2)	70.3-103 (10.2-15.0)	30-150
Polyethylene			
• Low density (LDPE)	9.0-14.5 (1.3-2.1)	8.3-31.4 (1.2-4.55)	100-650
• High density (HDPE)	26.2-33.1 (3.8-4.8)	22.1-31.0 (3.2-4.5)	10-1200
• Ultrahigh molecular weight (UHMWPE)	21.4-27.6 (3.1-4.0)	38.6-48.3 (5.6-7.0)	350-525
Polyethylene terephthalate (PET)	59.3 (8.6)	48.3-72.4 (7.0-10.5)	30-300
Polymethyl methacrylate (PMMA)	53.8-73.1 (7.8-10.6)	48.3-72.4 (7.0-10.5)	2.0-5.5
Polypropylene (PP)	31.0-37.2 (4.5-5.4)	31.0-41.4 (4.5-6.0)	100-600
Polystyrene (PS)	—	35.9-51.7 (5.2-7.5)	1.2-2.5
Polytetrafluoroethylene (PTFE)	—	20.7-34.5 (3.0-5.0)	200-400
Polyvinyl chloride (PVC)	40.7-44.8 (5.9-6.5)	40.7-51.7 (5.9-7.5)	40-80
<b>FIBER MATERIALS</b>			
Aramid (Kevlar 49)	—	3600-4100 (525-600)	2.8
Carbon (PAN precursor)			
• Standard modulus (longitudinal)	—	3800-4200 (550-610)	2
• Intermediate modulus (longitudinal)	—	4650-6350 (675-920)	1.8
• High modulus (longitudinal)	—	2500-4500 (360-650)	0.6
E Glass	—	3450 (500)	4.3
<b>COMPOSITE MATERIALS</b>			
Aramid fibers-epoxy matrix (aligned, $V_f = 0.6$ )			
• Longitudinal direction	—	1380 (200)	1.8
• Transverse direction	—	30 (4.3)	0.5
High modulus carbon fibers-epoxy matrix (aligned, $V_f = 0.6$ )			
• Longitudinal direction	—	760 (110)	0.3
• Transverse direction	—	28(4)	0.4
E glass fibers-epoxy matrix (aligned, $V_f = 0.6$ )			
• Longitudinal direction	—	1020 (150)	2.3
• Transverse direction	—	40 (5.8)	0.4
Wood			
• Douglas fir (12% moisture)			
Parallel to grain	—	108 (15.6)	—
Perpendicular to grain	—	2.4 (0.35)	—
• Red oak (12% moisture)			
Parallel to grain	—	112 (16.3)	—
Perpendicular to grain	—	7.2 (1.05)	—

<sup>a</sup> The strength of graphite, ceramics, and semiconducting materials is taken as flexural strength.

<sup>b</sup> The strength of concrete is measured in compression.

<sup>c</sup> Flexural strength value at 50% fracture probability.

**Table A.6 Mechanical Properties (in Tension) for Main Classes of Materials (Room-Temperature Conditions)**

Material	Yield Strength		Tensile Strength		Ductility, %EL [in 50mm (2in.)] <sup>a</sup>
	MPa	ksi	MPa	ksi	
1	2	3	4	5	6
<b>Metal Alloys<sup>b</sup></b>					
Molybdenum	565	82	655	95	35
Titanium	450	65	520	75	25
Steel (1020)	180	26	380	55	25
Nickel	138	20	480	70	40
Iron	130	19	262	38	45
Brass (70 Cu-30 Zn)	75	11	300	44	68
Copper	69	10	200	29	45
Aluminum	35	5	90	13	40
<b>Ceramic Materials<sup>c</sup></b>					
Zirconia (ZrO <sub>2</sub> ) <sup>d</sup>	—	—	800-1500	115-215	—
Silicon nitride (Si <sub>3</sub> N <sub>4</sub> )	—	—	250-1000	35-145	—
Aluminum oxide (Al <sub>2</sub> O <sub>3</sub> )	—	—	275-700	40-100	—
Silicon carbide (SiC)	—	—	100-820	15-120	—
Glass-ceramic (Pyroceram)	—	—	247	36	—
Mullite (3Al <sub>2</sub> O <sub>3</sub> -2SiO <sub>2</sub> )	—	—	185	27	—
Spinel (MgAl <sub>2</sub> O <sub>4</sub> )	—	—	110-245	16-36	—
Fused silica (SiO <sub>2</sub> )	—	—	110	16	—
Magnesium oxide (MgO) <sup>e</sup>	—	—	105	15	—
Soda-lime glass	—	—	69	10	—
<b>Polymers</b>					
Nylon 6,6	44.8-82.8	6.5-12	75.9-94.5	11.0-13.7	15-300
Polycarbonate (PC)	62.1	9.0	62.8-72.4	9.1-10.5	110-150
Polyester (PET)	59.3	8.6	48.3-72.4	7.0-10.5	30-300
Polymethyl methacrylate (PMMA)	53.8-73.1	7.8-10.6	48.3-72.4	7.0-10.5	2.0-5.5
Polyvinyl chloride (PVC)	40.7-44.8	5.9-6.5	40.7-51.7	5.9-7.5	40-80
Phenol-formaldehyde	—	—	34.5-62.1	5.0-9.0	1.5-2.0
Polystyrene (PS)	—	—	35.9-51.7	5.2-7.5	1.2-2.5
Polypropylene (PP)	31.0-37.2	4.5-5.4	31.0-41.4	4.5-6.0	100-600
Polyethylene – high density (HDPE)	26.2-33.1	3.8-4.8	22.1-31.0	3.2-4.5	10-1200
Polytetrafluoroethylene (PTFE)	—	—	20.7-34.5	3.0-5.0	200-400
Polyethylene – low density (LDPE)	9.0-14.5	1.3-2.1	8.3-31.4	1.2-4.55	100-650

<sup>a</sup> For polymers, percent elongation at break.

<sup>b</sup> Property values are for metal alloys in an annealed state.

<sup>c</sup> The tensile strength of ceramic materials is taken as flexural strength.

<sup>d</sup> Partially stabilized with 3 mol% Y<sub>2</sub>O<sub>3</sub>.

<sup>e</sup> Sintered and containing approximately 5% porosity.

## A.6 Linear Coefficient of Thermal Expansion

**Table A.7 Linear Coefficient of Thermal Expansion Values for Various Engineering Materials (Room-Temperature Conditions)**

Material	Coefficient of Thermal Expansion	
	$10^{-6}(\text{°C})^{-1}$	$10^{-6}(\text{°F})^{-1}$
1	2	3
<b>METALS AND METAL ALLOYS</b>		
<b>Plain Carbon and Low Alloy Steels</b>		
Steel alloy A36	11.7	6.5
Steel alloy 102	11.7	6.5
Steel alloy 1040	11.3	6.3
Steel alloy 4140	12.3	6.8
Steel alloy 4340	12.3	6.8
<b>Stainless Steels</b>		
Stainless alloy 304	17.2	9.6
Stainless alloy 316	15.9	8.8
Stainless alloy 405	10.8	6.0
Stainless alloy 440A	10.2	5.7
Stainless alloy 17-7PH	11.0	6.1
<b>Cast Irons</b>		
Gray irons		
• Grade G1800	11.4	6.3
• Grade G3000	11.4	6.3
• Grade G4000	11.4	6.3
Ductile irons		
• Grade 60-40-18	11.2	6.2
• Grade 80-55-06	10.6	5.9
<b>Aluminum Alloys</b>		
Alloy 1100	23.6	13.1
Alloy 2024	22.9	12.7
Alloy 6061	23.6	13.1
Alloy 7075	23.4	13.0
Alloy 356.0	21.5	11.9
<b>Copper Alloys</b>		
C11000 (electrolytic tough pitch)	17.0	9.4
C17200 (beryllium-copper)	16.7	9.3

Table A.7 (continued)

1	2	3
C26000 (cartridge brass)	19.9	11.1
C36000 (free-cutting brass)	20.5	11.4
C71500 (copper-nickel, 30%)	16.2	9.0
C93200 (bearing bronze)	18.0	10.0
<b>Magnesium Alloys</b>		
Alloy AZ31B	26.0	14.4
Alloy AZ91D	26.0	14.4
<b>Titanium Alloys</b>		
Commercially pure (ASTM grade 1)	8.6	4.8
Alloy Ti-5Al-2.5Sn	9.4	5.2
Alloy Ti-6Al-4V	8.6	4.8
<b>Precious Metals</b>		
Gold (commercially pure)	14.2	7.9
Platinum (commercially pure)	9.1	5.1
Silver (commercially pure)	19.7	10.9
<b>Refractory Metals</b>		
Molybdenum (commercially pure)	4.9	2.7
Tantalum (commercially pure)	6.5	3.6
Tungsten (commercially pure)	4.5	2.5
<b>Miscellaneous Nonferrous Alloys</b>		
Nickel 200	13.3	7.4
Inconel 625	12.8	7.1
Monel 400	13.9	7.7
Haynes alloy 25 Invar	12.3 1.6	6.8 0.9
Super invar	0.72	0.40
Kovar	5.1	2.8
Chemical lead	29.3	16.3
Antimonial lead (6%)	27.2	15.1
Tin (commercially pure)	23.8	13.2
Lead-Tin solder (60Sn-40Pb)	24.0	13.3
Zinc (commercially pure)	23.0-32.5	12.7-18.1
Zirconium, reactor grade 702	5.9	3.3
<b>GRAPHITE, CERAMICS, AND SEMICONDUCTING MATERIALS</b>		
Aluminum oxide		
• 99.9% pure	7.4	4.1
• 96%	7.4	4.1
• 90%	7.0	3.9

Table A.7 (continued)

1	2	3
Concrete	10.0-13.6	5.6-7.6
Diamond (natural)	0.11-1.23	0.06-0.68
Gallium arsenide	5.9	3.3
Glass, borosilicate (Pyrex)	3.3	1.8
Glass, soda-lime	9.0	5.0
Glass ceramic (Pyroceram)	6.5	3.6
Graphite		
• Extruded	2.0-2.7	1.1-1.5
• Isostatically molded	2.2-6.0	1.2-3.3
Silica, fused	0.4	0.22
Silicon	2.5	1.4
Silicon carbide		
• Hot pressed	4.6	2.6
• Sintered	4.1	2.3
Silicon nitride		
• Hot pressed	2.7	1.5
• Reaction bonded	3.1	1.7
• Sintered	3.1	1.7
Zirconia, 3 mol% Y <sub>2</sub> O <sub>3</sub>	9.6	5.3
<b>POLYMERS</b>		
Elastomers		
• Butadiene-acrylonitrile (nitrile)	235	130
• Styrene-butadiene (SBR)	220	125
• Silicone	270	150
Epoxy	81-117	45-65
Nylon 6,6	144	80
Phenolic	122	68
Polybutylene terephthalate (PBT)	108-171	60-95
Polycarbonate (PC)	122	68
Polyester (thermoset)	100-180	55-100
Polyetheretherketone (PEEK)	72-85	40-47
Polyethylene		
• Low density (LDPE)	180-400	100-220
• High density (HDPE)	106-198	59-110
• Ultrahigh molecular weight (UHMWPE)	234-360	130-200
Polyethylene terephthalate (PET)	117	65
Polymethyl methacrylate (PMMA)	90-162	50-90

Table A.7 (finished)

1	2	3
Polypropylene (PP)	146-180	81-100
Polystyrene (PS)	90-150	50-83
Polytetrafluoroethylene (PTFE)	126-216	70-120
Polyvinyl chloride (PVC)	90-180	50-100
<b>FIBER MATERIALS</b>		
Aramid (Kevlar 49)		
• Longitudinal direction	-2.0	-1.1
• Transverse direction	60	33
Carbon (PAN precursor)		
• Standard modulus		
Longitudinal direction	-0.6	-0.3
Transverse direction	10.0	5.6
• Intermediate modulus		
Longitudinal direction	-0.6	-0.3
• High modulus		
Longitudinal direction	-0.5	-0.28
Transverse direction	7.0	3.9
E Glass	5.0	2.8
• Longitudinal direction	-4.0	-2.2
• Transverse direction	70	40
High modulus carbon fibers-epoxy matrix ( $V_f = 0.6$ )		
• Longitudinal direction	-0.5	-0.3
• Transverse direction	32	18
E glass fibers-epoxy matrix ( $V_f = 0.6$ )		
• Longitudinal direction	6.6	3.7
• Transverse direction	30	16.7
Wood		
• Douglas fir (12% moisture)		
Parallel to grain	3.8-5.1	2.2-2.8
Perpendicular to grain	25.4-33.8	14.1-18.8
• Red oak (12% moisture)		
Parallel to grain	4.6-5.9	2.6-3.3
Perpendicular to grain	30.6-39.1	17.0-21.7

## Appendix B Properties of Structural-Steel Shapes

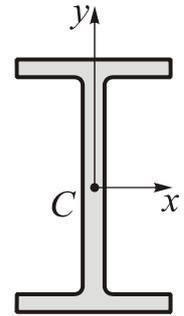
**Table B.1 Properties of Rolled-Steel (W) Shapes, Wide-Flange Sections (U.S. Customary Units)**

Source: The American Institute of Steel Construction (AISC).

**Note:** Axes  $x-x$  and  $y-y$  are principal centroidal axes.

\*A wide-flange shape is designated by the letter W followed by the nominal depth in inches and the weight in pounds per foot.

\*\*Value of  $W$  is sectional modulus of area.



Designation	Area in. <sup>2</sup>	Depth in.	Web thickness in.	Flange		Axis $x-x$			Axis $y-y$		
				Width in.	Thickness in.	$I$ in. <sup>4</sup>	$W$ in. <sup>3</sup>	$i$ in.	$I$ in. <sup>4</sup>	$W$ in. <sup>3</sup>	$i$ in.
				1	3	4	5	6	7	8	9
W 30 × 211	62.0	30.94	0.775	15.105	1.315	10300	663	12.9	757	100	3.49
W 30 × 132	38.9	30.31	0.615	10.545	1.000	5770	380	12.2	196	37.2	2.25
W 24 × 162	47.7	25.00	0.705	12.955	1.220	5170	414	10.4	443	68.4	3.05
W 24 × 104	30.6	24.06	0.500	12.750	0.750	3100	258	10.1	259		2.91
W 24 × 94	27.7	24.31	0.515	9.065	0.875	2700	222	9.87	109	24.0	1.98
W 24 × 84	24.7	24.10	0.470	9.020	0.770	2370	196	9.79	94.4		1.95
W 18 × 119	35.1	18.97	0.655	11.265	1.060	2190	231	7.90	253	44.9	2.69
W 18 × 106	31.1	18.73	0.590	11.200	0.940	1910	204	7.84	220		2.66
W 18 × 71	20.8	18.47	0.495	7.635	0.810	1170	127	7.50	60.3	15.8	1.70
W 18 × 50	14.7	17.99	0.355	7.495	0.570	800	88.9	7.38	40.1		1.65
W 18 × 35	10.3	17.70	0.300	6.000	0.425	510	57.6	7.04	15.3		1.22
W 16 × 100	29.4	16.97	0.585	10.425	0.985	1490	175	7.10	186	35.7	2.51
W 16 × 77	22.6	16.52	0.455	10.295	0.760	1110	134	7.00	138	26.9	2.47
W 16 × 57	16.8	16.43	0.430	7.120	0.715	758	92.2	6.72	43.1	12.1	1.60
W 16 × 40	11.8	16.01	0.305	6.995	0.505	518	64.7	6.63	28.9		1.57
W 16 × 31	9.12	15.88	0.275	5.525	0.440	375	47.2	6.41	12.4	4.49	1.17
W 14 × 145	42.7	14.78	0.680	15.500	1.090	1710	232	6.33	677		3.98
W 14 × 120	35.3	14.48	0.590	14.670	0.940	1380	190	6.24	495	67.5	3.74
W 14 × 82	24.1	14.31	0.510	10.130	0.855	882	123	6.05	148	29.3	2.48
W 14 × 53	15.6	13.92	0.370	8.060	0.660	541	77.8	5.89	57.7	14.3	1.92
W 14 × 26	7.69	13.91	0.255	5.025	0.420	245	35.3	5.65	8.91	3.54	1.08
W 12 × 87	25.6	12.53	0.515	12.125	0.810	740	118	5.38	241	39.7	3.07
W 12 × 72	21.1	12.25	0.430	12.040	0.670	597	97.4	5.31	195		3.04
W 12 × 50	14.7	12.19	0.370	8.080	0.640	394	64.7	5.18	56.3	13.9	1.96
W 12 × 35	10.3	12.50	0.300	6.560	0.520	285	45.6	5.25	24.5	7.47	1.54
W 12 × 14	4.16	11.91	0.200	3.970	0.225	88.6	14.9	4.62	2.36	1.19	0.753
W 10 × 60	17.6	10.22	0.420	10.080	0.680	341	66.7	4.39	116	23.0	2.57

Table B.1 (finished)

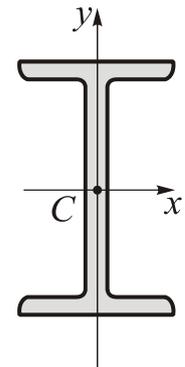
1	3	4	5	6	7	8	9	10	11	12	13
W 10 × 54	15.8	10.09	0.370	10.030	0.615	303	60.0	4.37	103		2.56
W 10 × 45	13.3	10.10	0.350	8.020	0.620	248	49.1	4.32	53.4	13.3	2.01
W 10 × 33	9.71	9.73	0.290	7.960	0.435	170	35.0	4.19	36.6		1.94
W 10 × 30	8.84	10.47	0.300	5.810	0.510	170	32.4	4.38	16.7	5.75	1.37
W 10 × 12	3.54	9.87	0.190	3.960	0.210	53.8	10.9	3.90	2.18	1.10	0.785
W 8 × 48	14.1	8.50	0.400	8.110	0.685	184	43.3	3.61	60.9		2.08
W 8 × 40	11.7	8.25	0.360	8.070	0.560	146	35.5	3.53	49.1		2.04
W 8 × 35	10.3	8.12	0.310	8.020	0.495	127	31.2	3.51	42.6	10.6	2.03
W 8 × 28	8.25	8.06	0.285	6.535	0.465	98.0	24.3	3.45	21.7	6.63	1.62
W 8 × 21	6.16	8.28	0.250	5.270	0.400	75.3	18.2	3.49	9.77	3.71	1.26
W 8 × 15	4.44	8.11	0.245	4.015	0.315	48.0	11.8	3.29	3.41	1.70	0.876
W 6 × 25	7.34	6.38	0.320	6.080	0.455	53.4	16.7	2.70	17.1		1.52
W 6 × 20	5.88	6.20	0.260	6.020	0.365	41.4	13.4	2.66	13.3		1.50
W 6 × 16	4.74	6.28	0.260	4.030	0.405	32.1	10.2	2.60	4.43		0.967
W 6 × 12	3.55	6.03	0.230	4.000	0.280	22.1	7.31	2.49	2.99		0.918

**Table B.2 Properties of Rolled-Steel (W) Shapes, Wide-Flange Sections (SI Units)**

**Note:** Axes  $x-x$  and  $y-y$  are principal centroidal axes.

\*A wide-flange shape is designated by the letter W followed by the nominal depth in millimeters and the mass in kilogram per meter.

\*\*Value of  $W$  is sectional modulus of area.

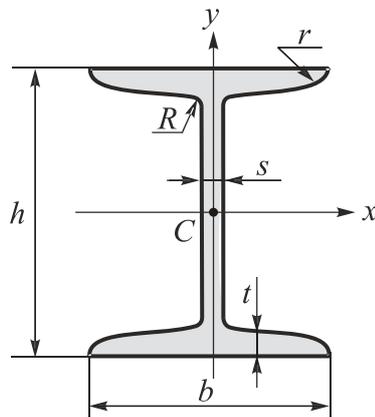


Designation*	Area, $10^3$ $\text{mm}^2$	Depth, mm	Flange		Web Thickness, mm	Axis $x-x$			Axis $y-y$	
			Width, mm	Thickness, mm		$I, 10^6$ $\text{mm}^4$	$i,$ mm	$W, 10^3$ $\text{mm}^3$	$I, 10^6$ $\text{mm}^4$	$i,$ mm
1	2	3	4	5	6	7	8	9	10	11
W 610 × 155	19.7	611	324	19.0	12.7	1290	256	4220	108	73.9
W 610 × 125	15.9	612	229	19.6	11.9	985	249	3220	39.3	49.7
W 460 × 158	20.1	476	284	23.9	15.0	795	199	3340	91.6	67.6
W 460 × 74	9.48	457	190	14.5	9.0	333	188	1457	16.7	41.9
W 460 × 52	6.65	450	152	10.8	7.6	212	179	942	6.4	31.0
W 410 × 114	14.6	420	261	19.3	11.6	462	178	2200	57.4	62.7
W 410 × 85	10.8	417	181	18.2	10.9	316	171	1516	17.9	40.6
W 410 × 60	7.61	407	178	12.8	7.7	216	168	1061	12	39.9
W 360 × 216	27.5	375	394	27.7	17.3	712	161	3800	282	101.1
W 360 × 122	15.5	363	257	21.7	13.0	367	154	2020	61.6	63.0
W 360 × 79	10.1	354	205	16.8	9.4	225	150	1271	24.0	48.8
W 310 × 107	13.6	311	306	17.0	10.9	248	135	1595	81.2	77.2

Table B.2 (finished)

1	2	3	4	5	6	7	8	9	10	11
W 310 × 74	9.48	310	205	16.3	9.4	164	132	1058	23.4	49.8
W 310 × 52	6.65	317	167	13.2	7.6	119	133	748	10.2	39.1
W 250 × 80	10.2	256	255	15.6	9.4	126	111	985	42.8	101.1
W 250 × 67	8.58	257	204	15.7	8.9	103	110	803	22.2	63.0
W 250 × 49	6.26	247	202	11.0	7.4	70.8	106	573	15.2	48.8
W 200 × 71	9.11	216	206	17.4	10.2	76.6	91.7	709	25.3	52.8
W 250 × 59	7.55	210	205	14.2	9.1	60.8	89.7	579	20.4	51.8
W 250 × 52	6.65	206	204	12.6	7.9	52.9	89.2	514	17.7	51.6
W 150 × 37	4.74	162	154	11.6	8.1	22.2	69	274	7.12	38.6
W 150 × 30	3.79	157	153	9.3	6.6	17.2	67.6	219	5.54	38.1
W 150 × 24	3.06	160	102	10.3	6.6	13.4	66	167	1.84	24.6
W 150 × 18	2.29	153	102	7.1	5.8	9.2	63.2	120	1.25	23.3

**Table B.3 Properties of Rolled-Steel Shapes (I-beam sections), USSR Standard (GOST 8239-72)**



$h$  – height of a beam,  
 $b$  – width of a flange,  
 $s$  – thickness of a web,  
 $t$  – average thickness of a flange,  
 $I$  – axial moment of inertia,  
 $W$  – sectional modulus,  
 $i$  – radius of gyration,  
 $S_x^T$  – first moment of a half-section.

Designation (number)	Dimensions, mm				Area, cm <sup>2</sup>	$I_x$ , cm <sup>4</sup>	$W_x$ , cm <sup>3</sup>	$i_x$ , cm	$S_x^T$ , cm <sup>3</sup>	$I_y$ , cm <sup>4</sup>	$W_y$ , cm <sup>3</sup>	$i_y$ , cm	Mass per meter, kg
	$h$	$b$	$s$	$t$									
1	2	3	4	5	6	7	8	9	10	11	12	13	14
10	100	55	4,5	7,2	12,0	198	39,7	4,06	23,0	17,9	6,49	1,22	9,46
12	120	64	4,8	7,3	14,7	350	58,4	4,88	33,7	27,9	8,72	1,38	11,5
14	140	73	4,9	7,5	17,4	572	81,7	5,73	46,8	41,9	11,5	1,55	13,7
16	160	81	5,0	7,8	20,2	873	109	6,57	62,	58,6	14,5	1,70	15,9
18	180	90	5,1	8,1	23,4	1290	143	7,42	81,4	82,6	18,4	1,88	18,4
18a	180	100	5,1	8,3	25,4	1430	159	7,51	89,8	114	22,8	2,12	19,9
20	200	100	5,2	8,4	26,8	1840	184	8,28	104	115	23,1	2,07	21,0
20a	200	110	5,2	8,6	28,9	2030	203	8,37	114	155	28,2	2,32	22,7
22	220	110	5,4	8,6	30,6	2550	232	9,13	131	157	28,6	2,27	24,0
22a	220	120	5,4	8,9	32,8	2790	254	9,22	143	206	34,3	2,50	25,8
24	240	115	5,6	9,5	34,8	3460	289	9,97	163	198	34,5	2,37	27,3
24a	240	125	5,6	9,8	37,5	3800	317	10,1	178	260	41,6	2,63	29,4
27	270	125	6,0	9,8	40,2	5010	371	11,2	210	260	41,5	2,54	31,5
27a	270	135	6,0	10,2	43,2	5500	407	11,3	229	337	50,0	2,80	33,9

Table B.3 (finished)

1	2	3	4	5	6	7	8	9	10	11	12	13	14
30	300	135	6,5	10,2	46,5	7080	472	12,3	268	337	49,9	2,69	36,5
30a	300	145	6,5	10,7	49,9	7780	518	12,5	292	436	60,1	2,95	39,2
33	330	140	7	11,2	53,8	9840	597	13,5	389	419	69,9	2,79	42,2
36	360	145	7,5	12,3	61,9	13380	743	14,7	423	519	71,1	2,89	48,6
40	400	155	8,3	13	72,6	19062	953	16,2	545	667	86,1	3,03	57
45	450	160	9	14,2	84,7	27696	1231	18,1	708	808	101	3,09	66,5
50	500	170	10	15,2	100	39727	1589	19,9	919	1043	123	3,23	78,5
55	550	180	11	16,5	118	35962	2035	21,8	1181	1366	151	3,39	92,7
60	600	190	12	17,8	138	76806	2560	23,6	1481	1725	182	3,54	108

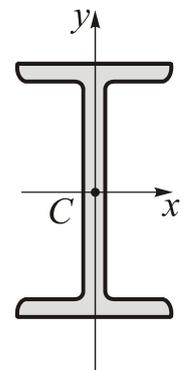
**Table B.4 Properties of Rolled-Steel (S) Shapes, American Standard I Beams (U.S. Customary Units)**

Source: The American Institute of Steel Construction (AISC).

**Note:** Axes  $x-x$  and  $y-y$  are principal centroidal axes.

\*An American Standard Beam is designated by the letter S followed by the nominal depth in inches and the weight in pounds per foot.

\*\*Value of  $W$  is sectional modulus of area.



Designation	Area	Depth	Web thickness	Flange		Axis $x-x$			Axis $y-y$		
				Width	Average thickness	$I$	$W$	$i$	$I$	$W$	$i$
1	3	4	5	6	7	8	9	10	11	12	13
S 24 × 100	29.3	24.00	0.745	7.245	0.870	2390	199	9.02	47.7	13.2	1.27
S 24 × 80	23.5	24.00	0.500	7.000	0.870	2100	175	9.47	42.2	12.1	1.34
S 20 × 95	27.9	20.00	0.800	7.200	0.916	1610	161	7.60	49.7	13.9	1.33
S 20 × 75	22.1	20.00	0.641	6.391	0.789	1280	128	7.60	29.6	9.32	1.16
S 18 × 70	20.6	18.00	0.711	6.251	0.691	926	103	6.71	24.1	7.72	1.08
S 18 × 54.7	16.1	18.00	0.461	6.001	0.691	804	89.4	7.07	20.8	6.94	1.14
S 15 × 50	14.7	15.00	0.550	5.640	0.622	486	64.8	5.75	15.7	5.57	1.03
S 15 × 42.9	12.6	15.00	0.411	5.501	0.622	447	59.6	5.95	14.4	5.23	1.07
S 12 × 50	14.7	12.00	0.687	5.477	0.659	305	50.8	4.55	15.7	5.74	1.03
S 12 × 35	10.3	12.00	0.428	5.078	0.544	229	38.2	4.72	9.87	3.89	0.980
S 10 × 35	10.3	10.00	0.594	4.944	0.491	147	29.4	3.78	8.36	3.38	0.901
S 10 × 25.4	7.46	10.00	0.311	4.661	0.491	124	24.7	4.07	6.79	2.91	0.954

Table B.4 (finished)

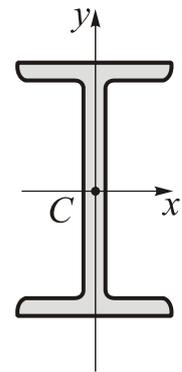
1	3	4	5	6	7	8	9	10	11	12	13
S 8 × 23	6.77	8.00	0.441	4.171	0.426	64.9	16.2	3.10	4.31	2.07	0.798
S 8 × 18.4	5.41	8.00	0.271	4.001	0.426	57.6	14.4	3.26	3.73	1.86	0.831
S 6 × 17.25	5.07	6.00	0.465	3.565	0.359	26.3	8.77	2.28	2.31	1.30	0.675
S 6 × 12.5	3.67	6.00	0.232	3.332	0.359	22.1	7.37	2.45	1.82	1.09	0.705
S 4 × 9.5	2.79	4.00	0.326	2.796	0.293	6.79	3.39	1.56	0.903	0.646	0.569
S 4 × 7.7	2.26	4.00	0.193	2.663	0.293	6.08	3.04	1.64	0.764	0.574	0.581

**Table B.5 Properties of Rolled-Steel (S) Shapes, American Standard I Beams (SI Units)**

*Note:* Axes  $x-x$  and  $y-y$  are principal centroidal axes.

\*An American Standard Beam is designated by the letter S followed by the nominal depth in millimeters and the mass in kilograms per meter.

\*\*Value of  $W$  is sectional modulus of area.



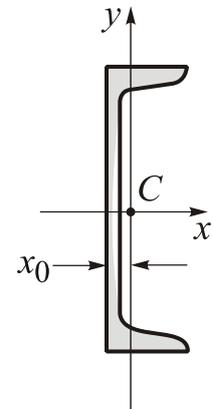
Designation*	Area, $10^3$ $\text{mm}^2$	Depth, mm	Flange		Web Thickness, mm	Axis $x-x$			Axis $y-y$	
			Width, mm	Thickness, mm		$I, 10^6$ $\text{mm}^4$	$i,$ mm	$W, 10^3$ $\text{mm}^3$	$I, 10^6$ $\text{mm}^4$	$i,$ mm
1	2	3	4	5	6	7	8	9	10	11
S 610 × 149	19.0	610	184	22.1	19.0	995	229	3260	19.9	32.3
× 119	15.2	610	178	22.1	12.7	878	241	2880	17.6	34.0
S 510 × 141	18.0	508	183	23.3	20.3	670	193	2640	20.7	33.8
× 112	14.3	508	162	20.1	16.3	533	193	2100	12.3	29.5
S 460 × 104	13.3	457	159	17.6	18.1	385	170	1685	10.0	27.4s
× 81	10.4	457	152	17.6	11.7	335	180	1466	8.66	29.0
S 380 × 74	9.5	381	143	15.8	14.0	202	146	1060	6.53	26.2
× 64	8.13	381	140	15.	10.4	186	151	977	5.99	27.2
S 310 × 74	9.48	305	139	16.8	17.4	127	116	833	6.53	26.2
× 52	6.64	305	129	13.8	10.9	95.3	120	625	4.11	24.9
S 250 × 52	6.64	254	126	12.5	15.1	61.2	96	482	3.48	22.9
× 38	4.81	254	118	12.5	7.9	51.6	103	406	2.83	24.2
S 200 × 34	4.37	203	106	10.8	11.2	27	78.7	266	1.79	20.3
× 27	3.5	203	102	10.8	6.9	24	82.8	236	1.55	21.1
S 150 × 26	3.27	152	90	9.1	11.8	11.0	57.9	144	0.96	17.2
× 19	2.36	152	84	9.1	5.8	9.20	62.2	121	0.76	17.9
S 100 × 14	1.80	102	70	7.4	8.3	2.83	39.6	55.5	0.38	14.5
× 11	1.45	102	67	7.4	4.8	2.53	41.6	49.6	0.32	14.8

**Table B.6 Properties of Rolled-Steel (C) Shapes, American Standard Channels (U.S. Customary Units)**

Source: The American Institute of Steel Construction (AISC).

**Notes:** Axes  $x-x$  and  $y-y$  are principal centroidal axes.The distance  $x_0$  is measured from the centroid to the back of the web.For axis  $y-y$ , the tabulated value of  $W$  is the smaller of the two sectional moduli for this axis.

\*An American Standard Channel designated by the letter C followed by the nominal depth in inches and the weight in pounds per foot.



Designation	Area in. <sup>2</sup>	Depth in.	Web thickness in.	Flange		Axis $x-x$			Axis $y-y$			
				Width in.	Average thickness in.	$I$ in. <sup>4</sup>	$W$ in. <sup>3</sup>	$i$ in.	$I$ in. <sup>4</sup>	$W$ in. <sup>3</sup>	$i$ in.	$x_0$ in.
1	2	3	4	5	6	7	8	9	10	11	12	13
C 15 × 50	14.7	15.00	0.716	3.716	0.650	404	53.8	5.24	11.0	3.78	0.867	0.798
C 15 × 40	11.8	15.00	0.520	3.520	0.650	349	46.5	5.44	9.23	3.37	0.886	0.777
C 15 × 33.9	9.96	15.00	0.400	3.400	0.650	315	42.0	5.62	8.13	3.11	0.904	0.787
C 12 × 30	8.82	12.00	0.510	3.170	0.501	162	27.0	4.29	5.14	2.06	0.763	0.674
C 12 × 25	7.35	12.00	0.387	3.047	0.501	144	24.1	4.43	4.47	1.88	0.780	0.674
C 12 × 20.7	6.09	12.00	0.282	2.942	0.501	129	21.5	4.61	3.88	1.73	0.799	0.698
C 10 × 30	8.82	10.00	0.673	3.033	0.436	103	20.7	3.42	3.94	1.65	0.669	0.649
C 10 × 25	7.35	10.00	0.520	2.886	0.436	91.2	18.2	3.52	3.36	1.48	0.676	0.617
C 10 × 20	5.88	10.00	0.379	2.739	0.436	78.9	15.8	3.66	2.81	1.32	0.692	0.606
C 10 × 15.3	4.49	10.00	0.240	2.600	0.436	67.4	13.5	3.87	2.28	1.16	0.713	0.634
C 8 × 18.75	5.51	8.00	0.487	2.527	0.390	44.0	11.0	2.82	1.98	1.01	0.599	0.565
C 8 × 13.75	4.04	8.00	0.303	2.343	0.390	36.1	9.03	2.99	1.53	0.854	0.615	0.553
C 8 × 11.5	3.38	8.00	0.220	2.260	0.390	32.6	8.14	3.11	1.32	0.781	0.625	0.571
C 6 × 13	3.83	6.00	0.437	2.157	0.343	17.4	5.80	2.13	1.05	0.642	0.525	0.514
C 6 × 10.5	3.09	6.00	0.314	2.034	0.343	15.2	5.06	2.22	0.866	0.564	0.529	0.499
C 6 × 8.2	2.40	6.00	0.200	1.920	0.343	13.1	4.38	2.34	0.693	0.492	0.537	0.511
C 4 × 7.25	2.13	4.00	0.321	1.721	0.296	4.59	2.29	1.47	0.433	0.343	0.450	0.459
C 4 × 5.4	1.59	4.00	0.184	1.584	0.296	3.85	1.93	1.56	0.319	0.283	0.449	0.457

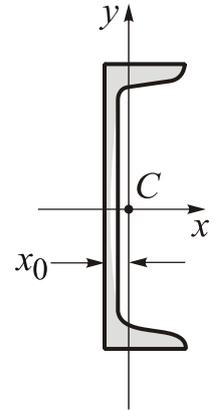
**Table B.7 Properties of Rolled-Steel (C) Shapes, American Standard Channel,\* (SI Units)**

**Notes:** Axes  $x$ - $x$  and  $y$ - $y$  are principal centroidal axes.

The distance  $x_0$  is measured from the centroid to the back of the web.

For axis  $y$ - $y$ , the tabulated value of  $W$  is the smaller of the two sectional moduli for this axis.

\*An American Standard Channel is designated by the letter C followed by the nominal depth in millimeters and the mass in kilograms per meter.

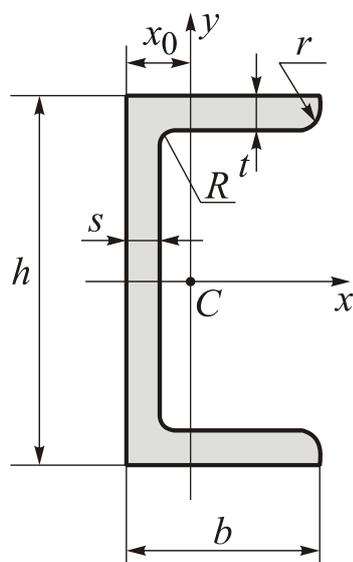


Designation"	Area, $10^3$ $\text{mm}^2$	Depth, mm	Flange		Web	Axis $x$ - $x$			Axis $y$ - $y$		
			Width, mm	Thickness, mm	Thickness, mm	$I$ , $10^6$ $\text{mm}^4$	$i$ , mm	$W$ , $10^3\text{mm}^3$	$I$ , $10^6$ mm	$i$ , mm	$x_0$ , mm
1	2	3	4	5	6	7	8	9	10	11	12
C 380 × 74	9.48	381	94	16.5	18.2	168	133	883	4.58	22.0	20.3
C 380 × 60	7.61	381	89	16.5	13.2	145	138	763	3.84	22.5	19.8
C 380 × 50	6.43	381	86	16.5	10.2	131	143	688	3.38	23.0	20.0
C 310 × 45	5.69	305	80	12.7	13	67.4	109	442	2.14	19.4	17.1
C 310 × 37	4.74	305	77	12.7	9.8	59.9	113	393	1.86	19.8	17.1
C 310 × 31	3.93	305	74	12.7	7.2	53.5	117	352	1.62	20.3	17.7
C 250 × 45	5.69	254	76	11.1	17.1	42.9	86.9	338	1.64	17.0	16.5
C 250 × 37	4.74	254	73	11.1	13.4	38.0	89.4	299	1.40	17.2	15.7
C 250 × 30	3.78	254	69	11.1	9.6	32.8	93	258	1.17	17.6	15.4
C 200 × 28	3.56	203	64	9.9	12.4	18.3	71.6	180	0.82	15.2	14.4
C 200 × 21	2.61	203	59	9.9	7.7	15.0	75.9	148	0.64	15.6	14.1
C 200 × 17	2.18	203	57	9.9	5.6	13.6	79.0	134	0.55	15.9	14.5
C 150 × 19	2.47	152	54	8.7	11.1	7.24	54.1	95.3	0.44	13.3	13.1
C 150 × 16	1.99	152	51	8.7	8.0	6.33	56.4	83.3	0.36	13.4	12.7
C 150 × 12	1.55	152	48	8.7	5.1	5.45	59.4	71.7	0.29	13.6	13.0
C 100 × 11	1.37	102	43	7.5	8.2	1.91	37.3	37.5	0.18	11.4	11.7
C 100 × 8	1.03	102	40	7.5	4.7	1.60	39.6	31.4	0.13	11.4	11.6

**Table B.8 Properties of Rolled-Steel (C) Shapes, USSR Standard (GOST 8240-72)**

**Notes:** Axes  $x$ - $x$  and  $y$ - $y$  are principal centroidal axes.

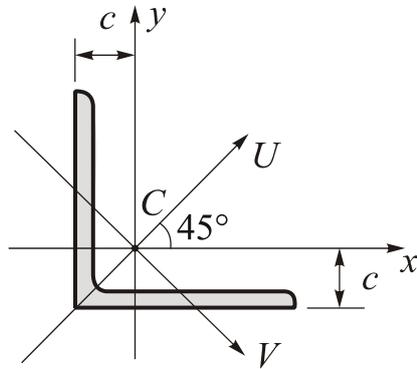
For axis  $y$ - $y$ , the tabulated value of  $W$  is the smaller of the two sectional moduli for this axis.



$h$  – height of a beam,  
 $b$  – width of a flange,  
 $s$  – thickness of a web,  
 $t$  – average thickness of a flange,  
 $W$  – sectional modulus,  
 $i$  – radius of gyration,  
 $S_x^T$  – first moment of a half-section,  
 $I$  – moment of inertia,  
 $x_0$  – distance from the centroid to the back of the web.

Designation (number)	Dimensions, mm				Area, cm <sup>2</sup>	$I_x$ , cm <sup>4</sup>	$W_x$ , cm <sup>3</sup>	$i_x$ , cm	$S_x^T$ , cm <sup>3</sup>	$I_y$ , cm <sup>4</sup>	$W_y$ , cm <sup>3</sup>	$i_y$ , cm	$x_0$ , cm
	$h$	$b$	$s$	$t$									
1	2	3	4	5	6	7	8	9	10	11	12	13	14
5	50	32	4,4	7	6,16	22,8	9,1	1,92	5,59	5,61	2,75	0,954	1,16
6,5	65	36	4,4	7,2	7,51	48,6	15	2,54	9	8,7	3,68	1,08	1,24
8	80	40	4,5	7,4	8,98	89,4	22,4	3,16	13,3	12,8	4,75	1,19	1,31
10	100	46	4,5	7,6	10,9	174	34,8	3,99	20,4	20,4	6,46	1,37	1,44
12	120	52	4,8	7,8	13,3	304	50,6	4,78	29,6	31,2	8,52	1,53	1,54
14	140	58	4,9	8,1	15,6	491	70,2	5,6	40,8	45,4	11	1,7	1,67
14a	140	62	4,9	8,7	17	545	77,8	5,66	45,1	57,5	13,3	1,84	1,87
16	160	64	5	8,4	18,1	747	93,4	6,42	54,1	63,3	13,8	1,87	1,8
16a	160	68	5	9	19,5	823	103	6,49	59,4	78,8	16,4	2,01	2
18	180	70	5,1	8,7	20,7	1090	121	7,24	69,8	86	17	2,04	1,94
18a	180	74	5,1	9,3	22,2	1190	132	7,32	76,1	105	20	2,18	2,13
20	200	76	5,2	9	23,4	1520	152	8,07	87,8	113	20,5	2,2	2,07
20a	200	80	5,2	9,7	25,2	1670	167	8,15	95,9	139	24,2	2,35	2,28
22	220	82	5,4	9,5	26,7	2110	192	8,89	110	151	25,1	2,37	2,21
22a	220	87	5,4	10,2	28,8	2330	212	8,99	121	187	30	2,55	2,46
24	240	90	5,6	10	30,6	2900	242	9,73	139	208	31,6	2,6	2,42
24a	240	95	5,6	10,7	32,9	3180	265	9,84	151	254	37,2	2,78	2,67
27	270	95	6	10,5	35,2	4160	308	10,9	178	262	37,3	2,73	2,47
30	300	100	6,5	11	40,5	5810	387	12	224	327	43,6	2,84	2,52
33	330	105	7	11,7	46,5	7980	484	13,1	281	410	51,8	2,97	2,59
36	360	110	7,5	12,6	53,4	10820	601	14,2	350	513	61,7	3,1	2,68
40	400	115	8	13,5	61,5	15220	761	15,7	444	642	73,4	3,26	2,75

**Table B.9 Properties of Rolled-Steel (L) Shapes, Angles with Equal Legs (U.S. Customary Units)** Source: The American Institute of Steel Construction (AISC)



**Notes:** Axes  $x-x$  and  $y-y$  are centroidal axes parallel to the legs.

The distance  $c$  is measured from the centroid to the back of the legs.

Axes  $U-U$  and  $V-V$  are principal centroidal axes.

The moment of inertia for axis  $V-V$ , which is the smaller of the two principal moments of inertia, can be found from the equation  $I_{VV} = Ai_{\min}^2$ .

The moment of inertia for axis  $U-U$ , which is the larger of the two principal moments of inertia, can be found from the equation  $I_{UU} + I_{VV} = I_{xx} + I_{yy}$ .

Designation	Area	Axis $x-x$ and Axis $y-y$				Axis $V-V$
		$I$	$W$	$i$	$c$	$i_{\min}$
in.	in. <sup>2</sup>	in. <sup>4</sup>	in. <sup>3</sup>	in.	in.	in.
1	2	3	4	5	6	7
L 8 × 8 × 1	15.0	89.0	15.8	2.44	2.37	1.56
L 8 × 8 × 3/4	11.4	69.7	12.2	2.47	2.28	1.58
L 8 × 8 × 1/2	7.75	48.6	8.36	2.50	2.19	1.59
L 6 × 6 × 1	11.0	35.5	8.57	1.80	1.86	1.17
L 6 × 6 × 3/4	8.44	28.2	6.66	1.83	1.78	1.17
L 6 × 6 × 1/2	5.75	19.9	4.61	1.86	1.68	1.18
L 5 × 5 × 7/8	7.98	17.8	5.17	1.49	1.57	0.973
L 5 × 5 × 1/2	4.75	11.3	3.16	1.54	1.43	0.983
L 5 × 5 × 3/8	3.61	8.74	2.42	1.56	1.39	0.990
L 4 × 4 × 3/4	5.44	7.67	2.81	1.19	1.27	0.778
L 4 × 4 × 1/2	3.75	5.56	1.97	1.22	1.18	0.782
L 4 × 4 × 3/8	2.86	4.36	1.52	1.23	1.14	0.788
L 3 1/2 × 3 1/2 × 3/8	2.48	2.87	1.15	1.07	1.01	0.687
L 3 1/2 × 3 1/2 × 1/4	1.69	2.01	0.794	1.09	0.968	0.694
L 3 × 3 × 1/2	2.75	2.22	1.07	0.898	0.932	0.584
L 3 × 3 × 1/4	1.44	1.24	0.577	0.930	0.842	0.592

**Table B.10 Properties of Rolled-Steel (L) Shapes, Angles with Equal Legs (SI Units)**

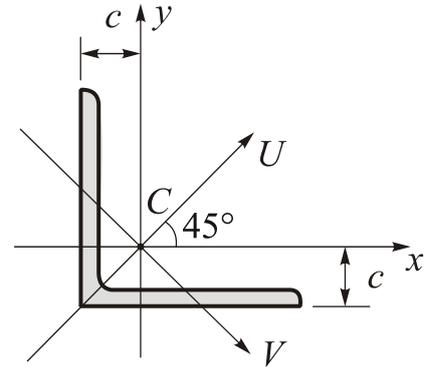
**Notes:** Axes  $x-x$  and  $y-y$  are centroidal axes parallel to the legs.

The distance  $c$  is measured from the centroid to the back of the legs.

Axes  $U-U$  and  $V-V$  are principal centroidal axes.

The moment of inertia for axis  $V-V$ , which is the smaller of the two principal moments of inertia, can be found from the equation  $I_{VV} = Ai_{\min}^2$ .

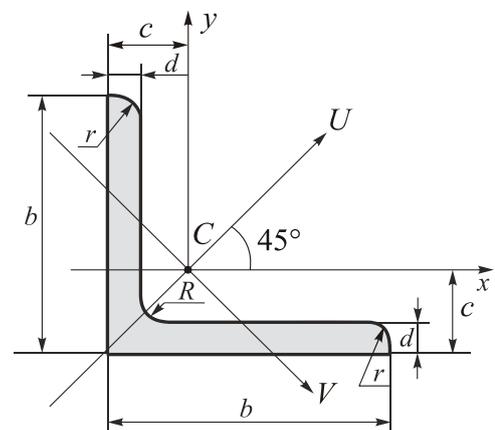
The moment of inertia for axis  $U-U$ , which is the larger of the two principal moments of inertia, can be found from the equation  $I_{UU} + I_{VV} = I_{xx} + I_{yy}$ .



Size and Thickness, mm	Area, $10^3$ mm <sup>2</sup>	Axis $x-x$ or $y-y$				Axis $V-V$
		$I, 10^6$ mm <sup>4</sup>	$i, mm$	$W, 10^3$ mm <sup>3</sup>	$c, mm$	$i_{\min}$ mm
1	2	3	4	5	6	7
L 203 × 203 × 25.4	9.68	37	61.8	259	60.2	39.6
L 203 × 203 × 19	7.36	29	62.8	200	57.9	40.1
L 203 × 203 × 12.7	5.0	20.2	63.6	137	55.6	40.4
L 152 × 152 × 25.4	7.1	14.8	45.6	140.4	47.2	29.7
L 152 × 152 × 15.9	4.59	10.1	46.8	92.8	43.9	30.0
L 152 × 152 × 9.5	2.8	6.41	47.8	57.8	41.7	30.2
L 127 × 127 × 19	4.48	6.53	38.2	74.2	38.6	24.8
L 127 × 127 × 12.7	3.07	4.70	39.2	51.8	36.3	25.0
L 127 × 127 × 9.5	2.33	3.64	39.5	39.7	35.3	25.1
L 102 × 102 × 19	3.51	3.19	30.1	46.0	32.3	19.8
L 102 × 102 × 12.7	2.42	2.31	30.9	32.3	30.0	19.9
L 102 × 102 × 6.4	1.25	1.27	31.8	17.2	27.7	20.2
L 89 × 89 × 9.5	1.6	1.20	27.3	18.9	25.7	17.5
L 89 × 89 × 6.4	1.09	0.84	27.7	13.0	24.6	17.6
L 76 × 76 × 12.7	1.77	0.92	22.8	17.5	23.7	14.8
L 76 × 76 × 6.4	0.93	0.52	23.6	9.46	21.4	15.0

**Table B.11 Properties of Rolled-Steel (L) Shapes, Angles with Equal Legs. USSR Standard (GOST 8509-72)**

$b$  – width of web,  
 $d$  – thickness,  
 $I$  – moment of inertia,  
 $i$  – radius of gyration,  
 $c$  – distance to centroid.



**Notes:** Axes  $x-x$  and  $y-y$  are centroidal axes parallel to the legs.

Axes  $U-U$  and  $V-V$  are principal centroidal axes.

The moment of inertia for axis  $V-V$ , which is the smaller of the two principal moments of inertia, can be found from the equation  $I_{VV} = Ai_{\min}^2$ .

The moment of inertia for axis  $U-U$ , which is the larger of the two principal moments of inertia, can be found from the equation  $I_{UU} + I_{VV} = I_{xx} + I_{yy}$ .

Designation (number)	$b$	$d$	Area, $\text{cm}^2$	Axes						$c$ , cm
				$x-x$		$U-U$		$V-V$		
	$I_x$ , $\text{cm}^4$	$i_x$ , cm		$I_{U_{\max}}$ , $\text{cm}^4$	$i_{U_{\max}}$ , cm	$I_{V_{\min}}$ , $\text{cm}^4$	$i_{V_{\min}}$ , cm			
1	2	3	4	5	6	7	8	9	10	11
2	20	3	1,13	0,4	0,59	0,63	0,75	0,17	0,39	0,6
		4	1,46	0,5	0,58	0,78	0,73	0,22	0,38	0,64
2,5	25	3	1,43	0,81	0,75	1,29	0,95	0,34	0,49	0,73
		4	1,86	1,03	0,74	1,62	0,93	0,44	0,48	0,76
2,8	28	3	1,62	1,16	0,85	1,84	1,07	0,48	0,55	0,8
3,2	32	3	1,86	1,77	0,97	2,8	1,23	0,74	0,63	0,89
		4	2,43	2,26	0,96	3,58	1,21	0,94	0,62	0,94
3,6	36	3	2,1	2,56	1,1	4,06	1,39	1,06	0,71	0,99
		4	2,75	3,29	1,09	5,21	1,38	1,36	0,7	1,04
4	40	3	2,35	3,55	1,23	5,63	1,55	1,47	0,79	1,09
		4	3,08	4,58	1,22	7,26	1,53	1,9	0,78	1,13
		5	3,79	5,53	1,2	8,75	1,54	2,3	0,79	1,17
4,5	45	3	2,65	5,13	1,39	8,13	1,75	2,12	0,89	1,21
		4	3,48	6,63	1,38	10,5	1,74	2,74	0,89	1,26
5	50	5	4,29	8,03	1,37	12,7	1,72	3,33	0,88	1,3
		3	2,96	7,11	1,55	11,3	1,95	2,95	1	1,33
		4	3,89	9,21	1,54	14,6	1,94	3,8	0,99	1,38
5,6	56	5	4,8	11,2	1,53	17,8	1,92	4,63	0,98	1,42
		4	4,38	13,1	1,73	20,8	2,18	5,41	1,11	1,52
		5	5,41	16	1,72	25,4	2,16	6,59	1,1	1,57
6,3	63	4	4,96	18,9	1,95	29,9	2,45	7,81	1,25	1,69
		5	6,13	23,1	1,94	36,6	2,44	9,52	1,25	1,74
		6	7,28	27,1	1,93	42,9	2,43	11,2	1,24	1,78
7	70	4,5	6,2	29	2,16	46	2,72	12	1,39	1,88
		5	6,86	31,9	2,16	50,7	2,72	13,2	1,39	1,9
		6	8,15	37,6	2,15	59,6	2,71	15,5	1,38	1,94
		7	9,42	43	2,14	68,2	2,69	17,8	1,37	1,99
		8	10,7	48,2	2,13	76,4	2,68	20	1,37	2,02
7,5	75	5	7,39	39,5	2,31	62,6	2,91	16,4	1,49	2,02
		6	8,78	46,6	2,3	73,9	2,9	19,3	1,48	2,06
		7	10,1	53,3	2,29	84,6	2,89	22,1	1,48	2,1
		8	11,5	59,8	2,28	94,6	2,87	24,8	1,47	2,15
8	80	9	12,8	66,1	2,27	105	2,86	27,5	1,46	2,18
		5,5	8,63	52,7	2,47	83,6	3,11	21,8	1,59	2,17
		6	9,38	57	2,47	90,4	3,11	23,5	1,58	2,19

Table B.11 (finished)

1	2	3	4	5	6	7	8	9	10	11
		7	10,8	65,3	2,45	104	3,09	27	1,58	2,23
		8	12,3	73,4	2,44	116	3,08	30,3	1,57	2,27
9	90	6	10,6	82,1	2,78	130	3,5	34	1,79	2,43
		7	12,3	94,3	2,77	150	3,49	38,9	1,78	2,47
		8	13,9	106	2,76	168	3,48	43,8	1,77	2,51
		9	15,6	118	2,75	186	3,46	48,6	1,77	2,55
10	100	6,5	12,8	122	3,09	193	3,88	50,7	1,99	2,68
		7	13,8	131	3,08	207	3,88	54,2	1,98	2,71
		8	15,6	147	3,07	233	3,87	60,9	1,98	2,75
		10	19,2	179	3,05	284	3,84	74,1	1,96	2,83
		12	22,8	209	3,03	331	3,81	86,9	1,95	2,91
		14	26,3	237	3	375	3,78	99,3	1,94	2,99
		16	29,7	264	2,98	416	3,74	112	1,94	3,06
11	110	7	15,2	176	3,4	279	4,29	72,7	2,19	2,96
		8	17,2	198	3,39	315	4,28	81,8	2,18	3
12,5	125	8	19,7	294	3,87	467	4,87	122	2,49	3,36
		9	22	327	3,86	520	4,86	135	2,48	3,4
		10	24,3	360	3,85	571	4,84	149	2,47	3,45
		12	28,9	422	3,82	670	4,82	174	2,46	3,53
		14	33,4	482	3,8	764	4,78	200	2,45	3,61
		16	37,8	539	3,78	853	4,75	224	2,44	3,68
14	140	9	24,7	466	4,34	739	5,47	192	2,79	3,78
		10	27,3	512	4,33	814	5,46	211	2,78	3,82
		12	32,5	602	4,31	957	5,43	248	2,76	3,9
16	160	10	31,4	774	4,96	1229	6,25	319	3,19	4,3
		11	34,4	844	4,95	1341	6,24	348	3,18	4,35
		12	37,4	913	4,94	1450	6,23	376	3,17	4,39
		14	43,3	1046	4,92	1662	6,2	431	3,16	4,47
		16	49,1	1175	4,89	1866	6,17	485	3,14	4,55
		18	54,8	1299	4,87	2061	6,13	537	3,13	4,63
		20	60,4	1419	4,85	2248	6,1	589	3,12	4,7
18	180	11	38,8	1216	5,6	1133	7,06	500	3,59	4,85
		12	42,2	1317	5,59	2093	7,04	540	3,58	4,89
20	200	12	47,1	1823	6,22	2896	7,84	749	3,99	5,37
		13	50,9	1961	6,21	3116	7,83	805	3,98	5,42
		14	54,6	2097	6,2	3333	7,81	861	3,97	5,46
		16	62	2363	6,17	3755	7,78	970	3,96	5,54
		20	76,5	2871	6,12	4560	7,72	1182	3,93	5,7
		25	94,3	3466	6,06	5494	7,63	1438	3,91	5,89
		30	111,5	4020	6	6351	7,55	1688	3,89	6,07
22	220	14	60,4	2814	6,83	1170	8,6	1159	4,38	5,93
		16	68,6	3175	6,81	5045	8,58	1306	4,36	6,02
25	250	16	78,4	4717	7,76	7492	9,78	1942	4,98	6,75
		18	87,7	5247	7,73	8337	9,75	2158	4,96	6,83
		20	97	5765	7,71	9160	9,72	2370	4,94	6,91
		22	106,1	6270	7,69	9961	9,69	2579	4,93	7
		25	119,7	7006	7,65	11125	9,64	2887	4,91	7,11
		28	133,1	7717	7,61	12244	9,59	3190	4,89	7,23
		30	142	8177	7,59	12965	9,56	3389	4,89	7,31

**Table B.12 Properties of Rolled-Steel (L) Shapes, Angles with Unequal Legs (U.S. Customary Units)** Source: The American Institute of Steel Construction.

**Notes:** Axes  $x-x$  and  $y-y$  are centroidal axes parallel to the legs.

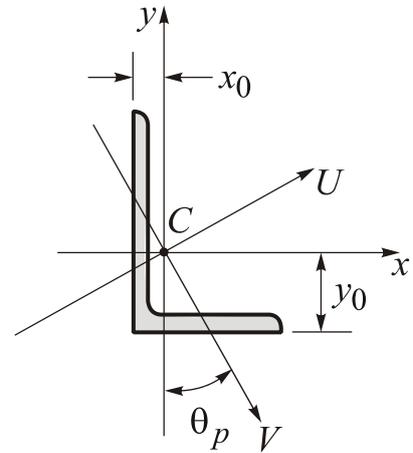
The distances  $x_0$  and  $y_0$  are measured from the centroid to the backs of the legs.

For axes  $x-x$  and  $y-y$ , the tabulated value of  $W$  is the smaller of the two sectional moduli for those axes.

Axes  $U-U$  and  $V-V$  are principal centroidal axes (accordingly, max and min).

The moment of inertia for axis  $V-V$ , which is the smaller of the two principal moments of inertia, can be found from the equation  $I_{VV} = Ai_{\min}^2$ .

The moment of inertia for axis  $U-U$ , which is the larger of the two principal moments of inertia, can be found from the equation  $I_{UU} + I_{VV} = I_{xx} + I_{yy}$ .



Designation, in.	Area in. <sup>2</sup>	Axis $x-x$				Axis $y-y$				Axis $V-V$	
		$I$ in. <sup>4</sup>	$W$ in. <sup>3</sup>	$i$ in.	$y_0$ in.	$I$ in. <sup>4</sup>	$W$ in. <sup>3</sup>	$i$ in.	$x_0$ in.	$i_{\min}$ in.	$\tan \theta_p$
1	2	3	4	5	6	7	8	9	10	11	12
L 8 × 6 × 1	13.00	80.8	15.1	2.49	2.65	38.8	8.92	1.73	1.65	1.28	0.543
L 8 × 6 × 1/2	6.75	44.3	8.02	2.56	2.47	21.7	4.79	1.79	1.47	1.30	0.558
L 7 × 4 × 3/4	7.69	37.8	8.42	2.22	2.51	9.05	3.03	1.09	1.01	0.860	0.324
L 7 × 4 × 1/2	5.25	26.7	5.81	2.25	2.42	6.53	2.12	1.11	0.917	0.872	0.335
L 6 × 4 × 3/4	6.94	24.5	6.25	1.88	2.08	8.68	2.97	1.12	1.08	0.860	0.428
L 6 × 4 × 1/2	4.75	17.4	4.33	1.91	1.99	6.27	2.08	1.15	0.987	0.870	0.440
L 6 × 4 × 3/8	3.61	13.5	3.32	1.93	4.90	1.94		1.17	0.877	0.941	0.446
L 5 × 3 1/2 × 3/4	5.81	13.9	4.28	1.55	1.75	5.55	2.22	0.977	0.996	0.748	0.464
L 5 × 3 1/2 × 1/2	4.00	9.99	2.99	1.58	1.66	4.05	1.56	1.01	0.906	0.755	0.479
L 5 × 3 × 1/2	3.75	9.45	2.91	1.59	1.75	2.58	1.15	0.829	0.750	0.648	0.357
L 5 × 3 × 1/4	1.94	5.11	1.53	1.62	1.66	1.44	0.614	0.861	0.657	0.663	0.371
L 4 × 3 1/2 × 1/2	3.50	5.32	1.94	1.23	1.25	3.79	1.52	1.04	1.00	0.722	0.750
L 4 × 3 1/2 × 1/4	1.81	2.91	1.03	1.27	1.16	2.09	0.808	1.07	0.909	0.734	0.759
L 4 × 3 × 1/2	3.25	5.05	1.89	1.25	1.33	2.42	1.12	0.864	0.827	0.639	0.543
L 4 × 3 × 3/8	2.48	3.96	1.46	1.26	1.28	1.92	0.866	0.879	0.782	0.644	0.551
L 4 × 3 × 1/4	1.69	2.77	1.00	1.28	1.24	1.36	0.599	0.896	0.736	0.651	0.558

Table B.12 (finished)

1	2	3	4	5	6	7	8	9	10	11	12
L 3 1/2 × 2 1/2 × 1/2	2.75	3.24	1.41	1.09	1.36	1.20		0.704	0.534	0.705	0.486
L 3 1/2 × 2 1/2 × 1/4	1.44	1.80	0.755	1.12	1.11	0.777		0.735	0.614	0.544	0.506
L 3 × 2 × 1/2	2.50	2.08	1.04	0.913	1.30	1.00		0.722	0.520	0.750	0.667
L 3 × 2 × 1/4	1.31	1.17	0.561	0.945	0.911	0.743		0.753	0.661	0.528	0.684
L 2 1/2 × 2 × 3/8	1.55	0.912	0.547	0.768	0.514	0.813		0.577	0.420	0.581	0.614
L 2 1/2 × 2 × 1/4	1.06	0.654	0.381	0.784	0.372	0.787		0.592	0.537	0.424	0.626

**Table B.13 Properties of Rolled-Steel (L) Shapes, Angles with Unequal Legs (SI Units)**

**Notes:** Axes  $x-x$  and  $y-y$  are centroidal axes parallel to the legs.

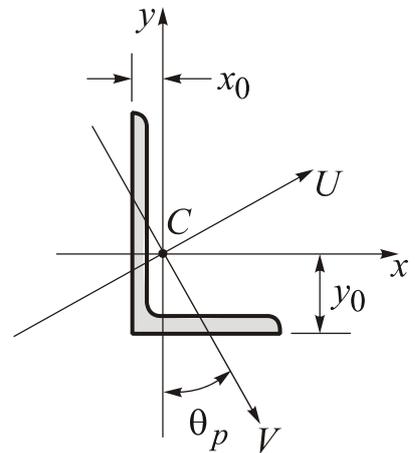
The distances  $x_0$  and  $y_0$  are measured from the centroid to the backs of the legs.

For axes  $x-x$  and  $y-y$ , the tabulated value of  $W$  is the smaller of the two sectional moduli for those axes.

Axes  $U-U$  and  $V-V$  are principal centroidal axes (accordingly, max and min).

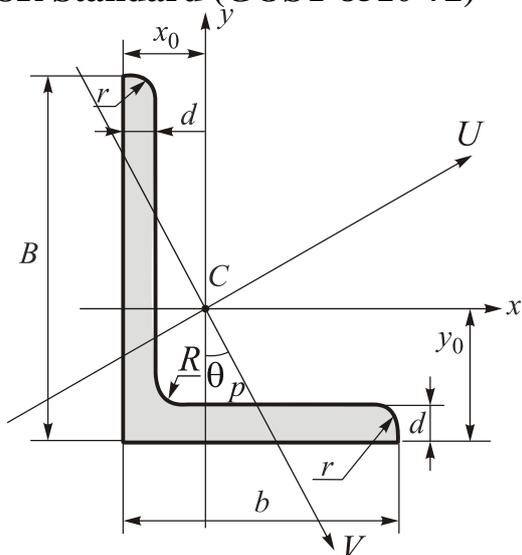
The moment of inertia for axis  $V-V$ , which is the smaller of the two principal moments of inertia, can be found from the equation  $I_{VV} = Ai_{\min}^2$ .

The moment of inertia for axis  $U-U$ , which is the larger of the two principal moments of inertia, can be found from the equation  $I_{UU} + I_{VV} = I_{xx} + I_{yy}$ .



Size and Thickness, mm	Area, $10^3 \text{ mm}^2$	Axis $x-x$				Axis $y-y$			Axis $V-V$	
		$I, 10^6 \text{ mm}^4$	$i, \text{ mm}$	$W, 10^3 \text{ mm}^3$	$y_0, \text{ mm}$	$I, 10^6 \text{ mm}^4$	$i, \text{ mm}$	$x_0, \text{ mm}$	$i_{\min}, \text{ mm}$	$\tan \theta_p$
1	2	3	4	5	6	7	8	9	10	11
L 203 × 152 × 25.4	8.39	33.6	63.3	247	67.3	16.2	43.9	41.9	32.5	0.543
L 203 × 152 × 12.7	4.35	18.4	65.1	131	62.7	9.03	45.6	37.3	33.0	0.558
L 152 × 102 × 19	4.48	10.2	47.7	102.4	52.8	3.61	28.4	27.4	21.8	0.428
L 152 × 102 × 9.5	2.33	5.62	49.1	54.4	49.3	2.04	29.6	23.9	22.3	0.446
L 127 × 76 × 12.7	2.42	3.93	40.3	47.7	44.5	1.07	21.1	19.1	16.5	0.357
L 127 × 76 × 6.4	1.25	2.13	41.2	25.1	42.2	0.60	21.9	16.7	16.8	0.371
L 102 × 76 × 12.7	2.10	2.10	31.6	31.0	33.8	1.01	21.9	21.0	16.2	0.543
L 102 × 76 × 6.4	1.09	1.15	32.	16.4	31.5	0.57	22.8	18.7	16.5	0.558
L 89 × 64 × 12.7	1.80	1.35	27.6	23.1	30.5	0.57	17.9	17.9	13.6	0.486
L 89 × 64 × 6.4	0.93	0.75	28.4	12.4	28.2	0.32	18.7	15.6	13.8	0.506
L 76 × 64 × 12.7	1.62	0.87	23.2	17.0	25.4	0.54	18.3	19.1	13.2	0.667
L 76 × 64 × 6.4	0.85	0.49	24.0	9.19	23.1	0.31	19.1	16.8	13.4	0.684
L 64 × 51 × 9.5	1.00	0.38	19.5	8.96	21.1	0.21	14.7	14.8	10.7	0.614
L 64 × 51 × 6.4	0.68	0.27	19.9	6.24	20.0	0.16	15.0	13.6	10.8	0.626

**Table B.14 Properties of Rolled-Steel (L) Shapes, Angles with Equal Legs. USSR Standard (GOST 8510-72)**



$B$  – width of larger leg,  
 $b$  – width of smaller leg,  
 $d$  – thickness of legs,  
 $I$  – moment of inertia,  
 $i$  – radius of gyration,  
 $x_0, y_0$  – distances from the centroid to the back of the legs.

Designation (number)	B mm	b mm	d mm	Area, cm <sup>2</sup>	Axes						Tan $\theta_p$		
					x-x		y-y		V-V			$x_0$ cm	$y_0$ cm
					$I_x$ cm <sup>4</sup>	$i_x$ cm	$I_y$ cm <sup>4</sup>	$i_y$ cm	$I_{Vmin}$ cm <sup>4</sup>	$i_{Vmin}$ cm			
1	2	3	4	5	6	7	8	9	10	11	12	13	14
2,5/1,6	25	16	3	1,16	0,7	0,78	0,22	0,44	0,13	0,34	0,42	0,86	0,392
3,2/2	32	20	3	1,49	1,52	1,01	0,46	0,55	0,28	0,43	0,49	1,08	0,382
			4	1,94	1,93	1	0,57	0,54	0,35	0,43	0,53	1,12	0,374
4/2,5	40	25	3	1,89	3,06	1,27	0,93	0,7	0,56	0,54	0,59	1,32	0,385
			4	2,47	3,93	1,26	1,18	0,69	0,71	0,54	0,63	1,37	0,381
4,5/2,8	45	28	3	2,14	4,41	1,43	1,32	0,79	0,79	0,61	0,64	1,47	0,382
			4	2,8	5,68	1,42	1,69	0,78	1,02	0,6	0,68	1,51	0,379
5/3,2	50	32	3	2,42	6,17	1,6	1,99	0,91	1,18	0,7	0,72	1,6	0,403
			4	3,17	7,98	1,59	2,56	0,9	1,52	0,69	0,76	1,85	0,401
5,6/3,6	56	36	4	3,58	11,4	1,78	3,7	1,02	2,19	0,78	0,84	1,82	0,406
			5	4,41	13,8	1,77	4,48	1,01	2,66	0,78	0,88	1,86	0,404
6,3/4,0	63	40	4	4,04	16,3	2,01	5,16	1,13	3,07	0,87	0,91	2,03	0,397
			5	4,98	19,9	2	6,26	1,12	3,72	0,86	0,95	2,08	0,396
			6	5,9	23,3	1,99	7,28	1,11	4,36	0,86	0,99	2,12	0,393
			8	7,68	29,6	1,96	9,15	1,09	5,58	0,85	1,07	2,2	0,386
7/4,5	70	45	5	5,59	27,8	2,23	9,05	1,27	5,34	0,98	1,05	2,28	0,406
7,5/5	75	50	5	6,11	34,8	2,39	12,5	1,43	7,24	1,09	1,17	2,39	0,436
			6	7,25	40,9	2,38	14,6	1,42	8,48	1,08	1,21	2,44	0,435
			8	9,47	52,4	2,35	18,5	1,4	10,9	1,07	1,29	2,52	0,43
8/5	80	50	5	6,36	41,6	2,56	12,7	1,41	7,58	1,09	1,13	2,6	0,387
			6	7,55	49	2,55	14,8	1,4	8,88	1,08	1,17	2,65	0,386
9/5,6	90	56	5,5	7,86	65,3	2,88	19,7	1,58	11,8	1,22	1,26	2,92	0,384
			6	8,54	70,6	2,88	21,2	1,58	12,7	1,22	1,28	2,95	0,384
			8	11,18	90,9	2,85	27,1	1,56	16,3	1,21	1,36	3,04	0,38

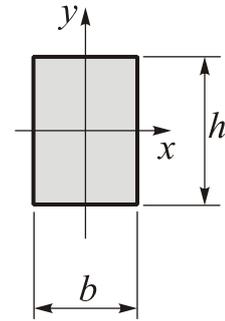
Table B.14 (finished)

1	2	3	4	5	6	7	8	9	10	11	12	13	
10/6,3	100	63	6	9,59	98,3	3,2	30,6	1,79	18,2	1,38	1,42	3,23	0,393
			7	11,1	113	3,19	35	1,78	20,8	1,37	1,46	3,28	0,392
			8	12,6	127	3,18	39,2	1,77	23,4	1,36	1,5	3,32	0,391
11/7	110	70	10	15,5	154	3,15	47,1	1,75	28,3	1,35	1,58	3,4	0,387
			6,5	11,4	142	3,53	45,6	2	26,9	1,53	1,58	3,55	0,402
			8	13,9	172	3,51	54,6	1,98	32,3	1,52	1,64	3,61	0,4
12,5/8	125	80	7	14,1	227	4,01	73,7	2,29	43,4	1,76	1,8	4,01	0,407
			8	16	256	4	83	2,28	48,8	1,75	1,84	4,05	0,406
			10	19,7	312	3,98	100	2,26	59,3	1,74	1,92	4,14	0,404
14/9	140	90	12	23,4	365	3,95	117	2,24	69,5	1,72	2	4,22	0,4
			8	18	364	4,49	120	2,58	70,3	1,98	2,03	4,49	0,411
			10	22,2	444	4,47	146	2,56	58,5	1,96	2,12	4,58	0,409
16/10	160	100	9	22,9	606	5,15	186	2,85	110	2,2	2,23	5,19	0,391
			10	25,3	667	5,13	204	2,84	121	2,19	2,28	5,23	0,39
			12	30	784	5,11	239	2,82	142	2,18	2,36	5,32	0,388
18/11	180	110	14	34,7	897	5,09	272	2,8	162	2,16	2,43	5,4	0,385
			10	28,3	952	5,8	276	3,12	165	2,42	2,44	5,88	0,375
			12	33,7	1123	5,77	324	3,1	194	2,4	2,52	5,97	0,374
20/12,5	200	125	11	34,9	1449	6,45	446	3,58	264	2,75	2,79	6,5	0,392
			12	37,9	1568	6,43	482	3,57	285	2,74	2,83	6,54	0,392
			14	43,9	1801	6,41	551	3,54	327	2,73	2,91	6,62	0,39
25/16	250	160	16	49,8	2026	6,38	617	3,52	367	2,72	2,99	6,71	0,388
			12	48,3	3147	8,07	1032	4,62	604	3,54	3,53	7,97	0,41
			16	63,6	4091	8,02	1333	4,58	781	3,5	3,69	8,14	0,408
			18	71,1	4545	7,99	1475	4,56	866	3,49	3,77	8,23	0,407
			20	78,5	4987	7,97	1613	4,53	949	3,48	3,85	8,31	0,405

### Appendix C Properties of Structural Lumber

**Table C.1 Properties of surfaced lumber**

*Note:* Axes  $x$ - $x$  and  $y$ - $y$  are principal centroidal axes.



Nominal dimensions $b \times h$	Net dimensions $b \times h$	Area $A = bh$	Axis $x$ - $x$		Axis $y$ - $y$		Weight per linear foot (weight density = 35 lb/ft <sup>3</sup> )
			Moment of inertia $I_x = \frac{bh^3}{12}$	Section modulus $W_x = \frac{bh^2}{6}$	Moment of inertia $I_y = \frac{hb^3}{12}$	Section modulus $W_y = \frac{hb^2}{6}$	
in.	in.	in. <sup>2</sup>	in. <sup>4</sup>	in. <sup>3</sup>	in. <sup>4</sup>	in. <sup>3</sup>	lb
2 × 4	1.5 × 3.5	5.25	5.36	3.06	0.98	1.31	1.3
2 × 6	1.5 × 5.5	8.25	20.80	7.56	1.55	2.06	2.0
2 × 8	1.5 × 7.25	10.88	47.63	13.14	2.04	2.72	2.6
2 × 10	1.5 × 9.25	13.88	98.93	21.39	2.60	3.47	3.4
2 × 12	1.5 × 11.25	16.88	177.98	31.64	3.16	4.22	4.1
3 × 4	2.5 × 3.5	8.75	8.93	5.10	4.56	3.65	2.1
3 × 6	2.5 × 5.5	13.75	34.66	12.60	7.16	5.73	3.3
3 × 8	2.5 × 7.25	18.13	79.39	21.90	9.44	7.55	4.4
3 × 10	2.5 × 9.25	23.13	164.89	35.65	12.04	9.64	5.6
3 × 12	2.5 × 11.25	28.13	296.63	52.73	14.65	11.72	6.8
4 × 4	3.5 × 3.5	12.25	12.51	7.15	12.51	7.15	3.0
4 × 6	3.5 × 5.5	19.25	48.53	17.65	19.65	11.23	4.7
4 × 8	3.5 × 7.25	25.38	111.15	30.66	25.90	14.80	6.2
4 × 10	3.5 × 9.25	32.38	230.84	49.91	33.05	18.89	7.9
4 × 12	3.5 × 11.25	39.38	415.28	73.83	40.20	22.97	9.6
6 × 6	5.5 × 5.5	30.25	76.3	27.7	76.3	111	7.4
6 × 8	5.5 × 7.5	41.25	193.4	51.6	104.0	37.8	10.0
6 × 10	5.5 × 9.5	52.25	393.0	82.7	131.7	47.9	12.7
6 × 12	5.5 × 11.5	63.25	697.1	121.2	159.4	58.0	15.4
8 × 8	7.5 × 7.5	56.25	263.7	70.3	263.7	70.3	13.7
8 × 10	7.5 × 9.5	71.25	535.9	112.8	334.0	89.1	17.3
8 × 12	7.5 × 11.5	86.25	950.5	165.3	404.3	107.8	21.0

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**Навчальне видання**

Деменко Владислав Федорович

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STRESS-STRAIN STATE AT A POINT OF ELASTIC DEFORMABLE SOLID

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Редактор Н.С. Дорошенко  
Технічний редактор Л.О. Кузьменко  
Комп'ютерна верстка: А.І. Безхутрий, А.А. Шевченко

Зв. план, 2010

Підписано до друку 07.07.10

Формат 60×84 1/16. Папір офс. № 2. Офс. друк

Ум. друк. арк. 16,2. Обл.-вид. арк. 18,25. Наклад 100 прим.

Замовлення . Ціна вільна

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Національний аерокосмічний університет ім. М.Є. Жуковського  
"Харківський авіаційний інститут"  
61070, Харків-70, вул. Чкалова, 17  
<http://www.khai.edu>  
Видавничий центр "ХАІ"  
61070, Харків-70, вул. Чкалова, 17  
[izdat@khai.edu](mailto:izdat@khai.edu)